Disturbance Estimation And Cancellation for Linear Uncertain Systems

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DISTURBANCE ESTIMATION AND CANCELLATION FOR LINEAR UNCERTAIN SYSTEMS

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Summary

This thesis was born on the boundary of the theory of control and a particular application, namely a smart structural system. More specifically, the primary motivation of this study comes from the following questions. Is it really possible to use smart structural system in a practical situation? What is the problem from the view point of the theory of control? In this work, it is assumed that one of the most important problems is that of robustness of the controlled system, and it is supposed that one of the solution would be to estimate uncertainty and disturbance without *a priori* knowledge in order to cancel their effect on system behaviour. This study provides a method, including its theoretical foundation, to estimate and cancel out any bounded disturbance and/or uncertainty.

One of the most important problems in control systems is the robustness of the controlled system. It is known that almost all physical systems, such as mechanical or structural system, contain some form of uncertainty. Even smart structural systems cannot escape from this problem. Such systems consist of host materials, sensing and actuating layers, which are attached or embedded to the host materials. The modelling of smart structural systems gives rise to infinite dimensional models if it were modelled by ordinary differential equations. In practice, however, a model is obtained by using the Finite Element Method, and, hence, a high order finite dimensional model is obtained. Such approximate models will inevitably generate uncertainty, representing unmodelled dynamics. In addition, such system models would suffer from parametric uncertainty and external disturbances. Thus, this type of system model would include many types of uncertainties.

In past decades, much research has been done using a deterministic approach for the robust control problem. The majority of this work assumes a known upper bound to uncertainty and disturbance, and robust controllers are determined deterministically. However, for smart structural systems, it may be difficult, or even impossible, to obtain such *a priori* knowledge of any disturbance. If this is the case, what can be said about the robustness of controlled system without *a priori* knowledge of any disturbance? Part of the answer to this question can be found in this thesis.

This thesis considers a linear uncertain system in which the uncertainty and/or disturbance is known to be bounded, but its bound is unknown. The main contribution is that an adaptive feedback control law is designed to estimate the bounded disturbance. The design of the adaptive control algorithm is novel and the adaptive control algorithm is easy to implement. This information can then be used to cancel the effect of the disturbance in the system. This has the advantage that, if further design objectives are to be realized, the controls can be designed based on the information from the known nominal model only

and not on the model with uncertainty.

This thesis is organized as follows. In Chapter 1, firstly, concept of stability of systems and deterministic approach of robust control are recalled. At the end of that chapter, it is implied that there is some limitation of that approach of robust control. In Chapter 2, motivated by the limitation discussed at Chapter 1, the method of disturbance estimation is introduced. Firstly, the statement of the problem is provided. It is followed by some preliminary works which are required for analysis. Then, for each class of systems, an adaptive algorithm, lemmas, theorem, and simulation examples are provided. The class of systems examined are second-order single-input linear systems, n^{th}order single-input linear systems, and multi-input linear systems. In Chapter 3, based on the works of Chapter 2, some applications of the method of disturbance estimation are presented. In Section 3.2, an adaptive algorithm which guarantees robustness of a controlled system is presented. In Section 3.3, treatment of input uncertainty and unmodelled dynamics is discussed. In the following section, Section 3.4, it is shown that under appropriate assumptions, it is possible to treat residual disturbance by the method proposed. In Section 3.5, the method is extended so that the method can be used only by outputs of a system. At last, in Section 3.6, it is demonstrated that parameter variations can be extracted from estimated disturbances. In Chapter 4, conclusion remarks and suggestions for future works are provided.

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Chapter 1

Deterministic approach of robust control and Lyapunov stabilization

1.1 Introduction

The aim of this chapter is to review the deterministic approach for robust control in the time domain and to discuss limitations of this approach.

For robust control of imperfectly known linear systems, much research has been done on the case when either parametric uncertainty, or uncertainty due to external disturbance is represented as a disturbance whose norm is assumed to be bounded by some known function ([3], [9], [16], [23], [24], and [37]). Under this assumption, often the resulting controller has a simple structure and the resulting closed-loop system is robust with respect to the uncertainty when using this simple controller. However, in some applications, this bounding information may be difficult, or impossible to obtain.

The outline of this chapter is as follows. Firstly, some mathematical terminology is defined and then, stability of a linear system and the concept of a Lyapunov function is recalled. Next, it is shown how, under appropriate assumptions, it is possible to stabilize a linear system with parametric uncertainty, and external disturbances using a deterministic approach. Finally, limitations of this deterministic approach for robust control are discussed. It is concluded that another deterministic method is required if the uncertainty or disturbance is bounded, but its bounding function is unknown.

1.2 Mathematical preliminaries

In this section, some terminologies, which are required for later lemmas and theorems, are defined.

Definition 1 $\mathbb{R}$ *denotes the set of real numbers.*

Definition 2 $\mathbb{R}^+$ *denotes the interval* $(0, \infty)$.

Definition 3 $\mathbb{R}_0^+$ *denotes the interval* $[0, \infty)$.

Definition 4 $\mathbb{N}$ *denotes the set of natural numbers.*

Definition 5 $\mathbb{R}^{m\times n}$, *where* $m, n \in \mathbb{N}$, *denotes the set of all real matrices of order* $m \times n$.

Definition 6 $\mathbb{R}^n$ *denotes the space of vectors with* n *components.*

Definition 7 $I \in \mathbb{R}^{n\times n}$ *is a identity matrix.*

Definition 8 A^t *denotes transpose of a matrix* A.

Definition 9 $\sigma_{\min}(A(t))$ *is a minimum eigenvalue of the matrix* $A(t)$ *at each fixed time* t.

Definition 10 $\sigma_{\max}(A(t))$ *is a maximum eigenvalue of the matrix* $A(t)$ *at each fixed time* t.

Definition 11 $diag(a_1, a_2, \cdots, a_n)$, *where* a_i *are scalars or submatrices in appropriate dimensions, denotes a matrix whose* i^{th} *row,* i^{th} *column element is* a_i *with* $i = 1, \cdots, n$ *and other elements are zero.*

Definition 12 *A matrix* A *is called positive definite matrix if all the eigenvalues of that matrix is positive.*

Definition 13 *A matrix* A *is called negative definite matrix if all the eigenvalues of that matrix is negative.*

Definition 14 $||x||$ *is a norm or 2-norm of a vector* $x = [x1 \;\; x2 \;\; \cdots \;\; x_n]^t$, *which is defined as follows:*

$$||x|| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \tag{1.2.1}$$

Definition 15 $||A||$ *is a induced-norm of the matrix* A *which is defined as follows:*

$$||A|| = \sqrt{\sigma_{\max}(A^t A)}.$$

Definition 16 $||A||_1$ *is a 1-norm of the matrix* A *which is defined as follows:*

$$||A||_1 = \sum_{i,j} |a_{ij}|,$$

where a_{ij} *is the element in the* i^{th} *row and* j^{th} *column of the matrix* A.

Definition 17 $\mathbb{B}(r)$ *denotes an open ball in* $\mathbb{R}^n$, *centred on the origin with radius* $r > 0$.

1.3 Stability of a system and Lyapunov function

In this section, the definition of a Lyapunov function and stability of a linear system are recalled.

Consider the linear time-varying system

$$\dot{x}(t) = A(t)x(t), \; x(t_0) = x_0, \; t \geq t_0, \tag{1.3.1}$$

where $x(t) \in \mathbb{R}^n$. It is assumed that $t \mapsto A(t) \in \mathbb{R}^{n\times n}$ is a continuous bounded matrix-valued function so that system (1.3.1) has existence and uniqueness of solutions.

1.3.1 Stability concepts

Loosely speaking, $x = 0$ is said to be a stable equilibrium point if each trajectory $x(t)$ of (1.3.1) remains 'close' to the state origin when x_0 is 'close' to the origin. More precise stability concepts are given below (Definition 5.4, 5.8, and 5.9 of [29]).

Definition 18 *The equilibrium point $x = 0$ is a stable equilibrium point of (1.3.1) if for all $t_0 \geq 0$ and $\epsilon > 0$, there exists $\delta(t_0, \epsilon) > 0$ such that*

$$||x_0|| < \delta(t_0, \epsilon) \Rightarrow ||x(t)|| < \epsilon \quad \forall t \geq t_0;$$

if δ is independent of t_0, then the equilibrium point is said to be uniformly stable.

A stronger type of stability, which requires that trajectories starting close to the state origin tend to the origin asymptotically, is given in definition 19.

Definition 19 *The equilibrium point $x = 0$ is globally asymptotically stable if*

1. *$x = 0$ is a stable equilibrium point, and*
2. *$\lim_{t\to\infty} x(t) = 0$ for all $x_0 \in \mathbb{R}^n$;*

If, in addition, $x = 0$ is uniformly stable and $x(t)$ converges uniformly to 0, that is there exists $\delta > 0$ and a function $\gamma(\tau, x) : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}_0^+$ such that

$$\lim_{\tau\to\infty} \gamma(\tau, x_0) = 0 \; \forall x_0 \in \mathbb{R}^n$$

and

$$||x_0|| < \delta \Rightarrow ||x(t)|| \leq \gamma(t - t_0, x_0) \quad \forall t \geq t_0,$$

then $x = 0$ is globally uniformly asymptotically stable.

Remark 1 For linear differential systems, there is no difference between asymptotic stability and global asymptotic stability, namely the two concepts are equivalent. Often, for linear systems, these properties are said to relate to the system, rather than the equilibrium point, since there is only one equilibrium point.

1.3.2 Lyapunov stability theorems

The so-called 'second method' of Lyapunov can be used to determine the stability properties of a system without explicitly obtaining the solutions to the differential equations defining system. Before stating the theorems, some definitions are required (Definition 5.11, 5.13, and 5.14 of [29]).

Definition 20 *A function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. If $\alpha \in \mathcal{K}$ and, in addition, $\alpha(r) \to \infty$ as $r \to \infty$, then α is said to belong to class $\mathcal{K}_\infty$.*

Definition 21 *A continuous function $(t, x) \mapsto V(t, x) : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}_0^+$ is called positive definite if, for some $\alpha \in \mathcal{K}_\infty$ and $\forall t \geq t_0$,*

1. *$V(t, 0) = 0$*

2. $V(t,x) \geq \alpha(||x||)\ \forall x \in \mathbb{R}^n$

Note that V is said to be negative definite if $-V$ is positive definite.

Sometimes a positive definite function is required to be bounded as t varies. This motivates the next definition.

Definition 22 *A continuous function $V : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}_0^+$ is said to be decrescent if there exists a function $\beta \in K$ such that*

$$V(t,x) \leq \beta(||x||)\ \forall x \in \mathbb{R}^n,\ t \geq 0$$

Remark 2 In view of Definition 22, $V(t,x)$ tends to zero uniformly with respect to t as $||x|| \to 0$.

Some Lyapunov stability theorems now follow (Theorem 5.16 of [29]). Loosely speaking, basic Lyapunov theorems state that if $V(t,x)$ is a positive definite function with continuous partial derivatives and, along solutions, $\dot{V}(t,x) \leq 0$, then one can conclude stability of the equilibrium point. The time derivative of V is taken along trajectories of (1.3.1), that is

$$\begin{aligned} \dot{V}(t,x) &:= \frac{dV(t,x)}{dt} \\ &= \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} A(t)x(t). \end{aligned}$$

Theorem 1 *If there exists a C^1, decrescent, positive definite function $V : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}_0^+$ such that, along solutions to (1.3.1), $\dot{V}(t,x) \leq 0$, then $x = 0$ is uniformly stable.*

Theorem 2 *If there exists a C^1, decrescent positive definite function $V : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}_0^+$ such that, along solutions to (1.3.1), $\dot{V}(t,x)$ is negative definite, then $x = 0$ is uniformly asymptotically stable.*

Remark 3 A function V that satisfies Theorem 1 or 2 is known as a *Lyapunov function.*

As it can be seen in the above theorems, the major advantage of Lyapunov-based stability analysis is its simplicity and its abstraction. However, its main disadvantage is that the theorems do not say how to find the Lyapunov functions.

With respect to advantages, since the only requirement for the stability of the system is to satisfy negative definiteness of the time derivative of a candidate Lyapunov function, it is easy to evaluate stability of an equilibrium point of a system. In fact, both the deterministic approach of robust control of linear uncertain systems, introduced in the next section and adaptive/robust control of linear uncertain systems, developed by this project, are based on the construction of a candidate Lyapunov function.

With respect to the main disadvantage, it is required to construct candidate Lyapunov functions for each class of systems examined. In general, it may be difficult to construct candidate Lyapunov functions for time-varying linear systems and nonlinear systems. However, for linear time-invariant systems, it is more straightforward, since, a quadratic candidate Lyapunov function can be used. For instance, consider the following linear time-invariant system:

$$\dot{x}(t) = Ax(t),\ x(t) \in \mathbb{R}^n. \tag{1.3.2}$$

Define the candidate Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_0^+$ by

$$V(x) := x^t P x$$

where $P \in \mathbb{R}^{n\times n}$ is a positive definite symmetric matrix. The time derivative of V is

$$\dot{V}(x(t)) = \dot{x}^t(t)Px(t) + x^t(t)P\dot{x}(t).$$

Evaluating the time derivative of V along solution to (1.3.2) gives

$$\begin{aligned}\dot{V}(x(t)) &= x^t(t)(A^tP + PA)x(t) \\ &= -x^t(t)Qx(t)\end{aligned} \tag{1.3.3}$$

where $-Q = A^tP + PA$ (see [21]). It is well known that x^tQx satisfies

$$\sigma_{\min}(Q)||x(t)||^2 \leq x^tQx \leq \sigma_{\max}(Q)||x(t)||^2$$

for any positive definite matrix Q and, therefore, it follows from (1.3.3) that, if $Q > 0$,

$$\dot{V}(x(t)) \leq -\sigma_{\min}(Q)||x(t)||^2.$$

Therefore, if there exists a symmetric positive definite matrix P such that Q is positive definite then it follows from Theorems 1 and 2 that the linear system (1.3.2) is stable.

1.4 Deterministic robust controllers

In this section, the deterministic approach to robust control of a linear system with uncertainty is reviewed (see [9], [16], [23], [24], and references therein.). Firstly, it is shown that parametric uncertainty, nonlinear effects, and/or external disturbance can be modelled as a time-varying disturbance to the system. Next, it is shown that, under certain assumptions, it is always possible to eliminate the effect of such a disturbance and, hence, the controlled system is robust with respect to uncertainty/disturbance in the system.

Consider the system:

$$\dot{x}(t) = (A + \Delta A(t))x(t) + f(t) + Bu(t) \tag{1.4.1}$$

where $x(t) \in \mathbb{R}^n$ represents state of system, $u(t) \in \mathbb{R}^m$ $(m \leq n)$ represents a control input to the system, $A \in \mathbb{R}^{n\times n}$ represents a known *nominal* term, $\Delta A \in \mathbb{R}^{n\times n}$ represents parametric uncertainty of the system, $B \in \mathbb{R}^{n\times m}$ represents the input matrix, and $f(t) \in \mathbb{R}^n$ represents nonlinear effects and/or an external disturbance to the system.

Assume the following relations hold.

$$\Delta A(t) = BE(t) \tag{1.4.2}$$

$$f(t) = Bw(t) \tag{1.4.3}$$

These relations are called *matching conditions* ([23] and [24]). Defining

$$p(t) := E(t) + w(t) \tag{1.4.4}$$

and substituting (1.4.2), (1.4.3), and (1.4.4) in (1.4.1) gives

$$\dot{x}(t) = Ax(t) + B(p(t) + u(t)) \tag{1.4.5}$$

Therefore, if matching conditions hold, then the effect of any uncertainty, including parametric uncertainty, nonlinear effect, and external disturbance can be regarded as an external disturbance to the system which can be directly influenced by the controller.

One may ask the question 'how can the effect of such a disturbance be eliminated?'. Over the last decades, many studies have been undertaken on this problem (see for example [3], [9], [16], [23], [24], [37], and references therein.). In this section, a deterministic approach for robust control of systems, initially used by Gutman, Leitmann, and Corless and Leitmann ([24], [23], [9], and [16]), is recalled. The outline of this approach is as follows.

1. It is assumed that an upper bound of the disturbance $p(t)$ is known, i.e. there is a known function $\rho(t)$ which satisfies $||p(t)|| \leq \rho(t)$.
2. Based on the knowledge of $\rho(t)$, a control input is used to force a trajectory of the system into a certain subspace.
3. If the nominal system is stable, then it can be shown that a trajectory of the system converges to the equilibrium point.

The simplest realization of such a control input is given by (see [23] for more details):

$$u(t) = -\frac{B^t P x(t)}{||B^t P x(t)||}\rho(t), \tag{1.4.6}$$

where $P \in \mathbb{R}^{n\times n}$ is a positive definite symmetric constant matrix. In (1.4.6), $B^t P x(t)$ is used to define the subspace $\mathcal{S} = \{x \in \mathbb{R}^n : B^t P x = 0\}$. Also, the sign of the term $B^t P x(t)$ is switched around the subspace $\mathcal{S}$. Moreover, the magnitude of this control input is always greater than or equal to the magnitude of the disturbance. Thus, using the controller, given by (1.4.6), a trajectory is forced into the subspace $\mathcal{S}$ and eventually converges to the equilibrium. In fact, it is possible to show this using a Lyapunov-based analysis. Define a candidate Lyapunov function by

$$V(x) := x^t P x,$$

where $P \in \mathbb{R}^n$ is a positive definite symmetric constant matrix. Taking the time derivative of V along solutions to (1.4.5) gives

$$\dot{V}(x(t)) = x^t(t)(A^t P + PA)x(t) + 2x^t(t)PB(u(t) + p(t)).$$

It is supposed that there exists positive definite symmetric matrix Q such that $A^t P + PA = -Q$ so that

$$\dot{V}(x(t)) = -x^t(t)Qx(t) + 2(B^t P x(t))^t(u(t) + p(t)) \tag{1.4.7}$$

In view of (1.4.6) and (1.4.7), it follows that

$$\begin{aligned}\dot{V}(x(t)) &= -x^t(t)Qx(t) + 2(B^t P x(t))^t\left(-\frac{B^t P x(t)}{||B^t P x(t)||}\rho(t) + p(t)\right)\\ &\leq -x^t(t)Qx(t)\\ &\leq -\sigma_{\min}(Q)||x(t)||^2\end{aligned}$$

Hence, using the controller, given by (1.4.6), the time derivative of V along solutions is negative definite. Thus, V is a Lyapunov function and, hence, by Theorem 2 it follows that, with this control input, system (1.4.5), is uniformly asymptotically stable despite the existence of uncertainty/disturbance in the system. Therefore, if *a priori* knowledge of the disturbance/uncertainty is known, then a system with matched uncertainty/disturbance is robustly controlled using (1.4.6).

Remark 4 In practice, the robust controller (1.4.6) cannot be used because the control input produced by this type of controller is a discontinuous signal. Thus, in practice, an appropriate continuous version of this controller is used. However, since the main objective of this chapter is to introduce the basic idea of a deterministic controller, the simplest type of controller have been chosen.

1.5 Limitation of deterministic approach

In the previous section, it is recalled that if *a priori* knowledge of the uncertainty/disturbance is known, it is possible to stabilize a system despite the existence of uncertainty/disturbance. However, the question is what can be achieved if such *a priori* knowledge of disturbance is not known? In practice, such a situation might happen. The robust controller given by (1.4.6) obviously cannot be used for such a situation. In such a situation, is there an alternative approach which synthesizes a robust controller for such a system? The answer is affirmative. The solution to this problem is given in later chapters.

1.6 Conclusions

In this section, a deterministic approach of robust control in the time domain is reviewed. Firstly, the concepts of stability and Lyapunov function are recalled. Then, a deterministic approach of robust control is reviewed. Finally, it is concluded that if *a priori* knowledge of disturbance is not known, an alternative method to control the behaviour of the system is required in order to ensure that the controlled system is robust with respect to the uncertainty in the system.

Chapter 2

Disturbance estimation and cancellation -the basis-

2.1 Introduction

The principal aim of this chapter is to describe and present a new approach to robust/adaptive control of imperfectly known linear systems.

As discussed in the previous chapter, many robust control problems are studied using a deterministic approach for robust control, which assumes prior knowledge of an uncertainty/disturbance bound or bounding function is known. In practice, however, it is supposed that such *a priori* knowledge of the uncertainty and/or disturbance may be difficult or almost impossible to obtain for the specific application.

Although there have been numerous studies in the area of robust control, there are very few studies that considers the problem of estimating and cancelling uncertainty and/or disturbance without *a priori* knowledge of uncertainty and/or disturbance. Those studies can be classified into three classes. One class consists of methods that use inverse dynamics of the nominal model to estimate the disturbance, see [19] for example. A second class of methods utilise observers with Lyapunov min-max type controllers, for example see [35]. The final class involves those methods that use high gain disturbance observers as studied in [32]. In the traditional disturbance observer (see [19], [13], and [25] for example), it is shown that the disturbance and uncertainty can be estimated using inverse dynamics of the nominal system. In the study on disturbance observers with Lyapunov min-max type controllers, ([36], [38] and [35]), it is shown that the uncertainty/disturbance can be estimated using an observer-like system with Lyapunov min-max type controllers, and an adaptive law. In the studies [36] and [38], it is shown that, with *a priori* knowledge of the bound on the uncertainty/disturbance, it is possible to estimate the uncertainty/disturbance using a non-adaptive control law. In [35], using an adaptive control law, it is shown that the uncertainty/disturbance can be estimated without *a priori* knowledge of the disturbance/uncertainty. In the study of the high-gain disturbance observers [32], the disturbance is treated as one of the states of the observer, and it was shown that, using a constant gain high gain observer, the disturbance can be estimated.

In the context of adaptive control, there are some studies which try to stabilize unknown systems using a control input, which is produced by an observer-like system (see [2], [5], and [18], for example.). However, they only consider parametric uncertainty for which all the parameters of the system are unknown. Moreover, those methods do not provide an estimation of the disturbance.

In addition, there are some studies that consider adaptive version of Lyapunov min-max type robust controllers, which do not require *a priori* knowledge of disturbance (see [8], [17], and [27] for example).

For this investigation, an on-line uncertainty/disturbance estimation and cancellation method is proposed. The approach is based on the inverse problem of tracking and it does not require any *a priori* knowledge on the bounds of any disturbances. The basic idea for estimating the disturbances is to track the state of the real system, to be controlled, by using the output of an observer-like system. The real system is assumed to be modelled as an additive unknown perturbation to a known linear system, known as the *nominal model*. The observer-like system is designed to be a known linear system with the same system matrices as the nominal system of the real system. Since both systems have the same system matrices, if it were possible to track the real system by the observer-like system, then the control input, for tracking, will produce almost exactly the same signal as the disturbance. In addition to this basic method, a feedforward filter is introduced in order to estimate and cancel out the disturbance in the closed-loop system. The resulting robust control scheme and the controller itself are very simple and hence, it is easy to implement in practice.

Although the method developed in this study and the disturbance observers, described above, share the same basic philosophy, they use different methods to that used in this investigation. Compared with the Lyapunov min-max controllers-based disturbance observer, the structure of the controller to the observer-like system is different. For this investigation, state feedback is used and for the Lyapunov min-max type controllers based observer, a Lyapunov min-max type controller is used. Compared with a high-gain disturbance observer, the way to express disturbance and adaptive law is different. For the high-gain disturbance observer, the disturbance is expressed as a state of the observer and, also, the gain is a constant and hence, non-adaptive. For this investigation, the disturbance is estimated using a control input to an observer-like system and the gain of the control input to the observer-like system is determined by a specified adaptation law.

The outline of this chapter is as follows. Firstly, a statement of the problem is presented and, in this section, a detailed explanation of disturbance estimation for both the open-loop system and the closed-loop system are introduced. It is shown that inverse tracking can be achieved for the open-loop system, but for the closed-loop system, a feedforward filter is required to estimate and cancel out the disturbance. In the next section, some preliminary works are presented. Then, in the following sections, for some classes of systems, algorithms, lemmas, and theorems for the estimation and cancellation of the disturbance, without any *a priori* knowledge, are presented. The classes of systems examined are second order single-input system, n^{th}order single-input system, and multi-input system. Also, for each class of system, simulation examples are shown to demonstrate the method developed.

2.2 Problem statement

In this section, the basic method of disturbance/uncertainty estimation and cancellation is introduced. Firstly, it is recalled that additive parametric uncertainty in a system can be modelled as an external disturbance to the system. Next, for the case when *a priori* knowledge of bounded function of uncertainty/disturbance is *unknown*, it is shown that an estimate of the disturbance for the open-loop system can be obtained using an inverse approach to tracking. Then a feedforward filter and modified reference signal is introduced and, finally, it is shown that the disturbance can be estimated and cancelled in the closed-loop system using the feedforward filter, modified reference signal and the inverse tracking problem approach.

Consider the following system:

$$\begin{aligned} \dot{x}(t) &= (A+\Delta A)x(t) + Bu(t) + v(t), \; t \geq t_0 \geq 0, && (2.2.1) \\ x(t_0) &= x^0, && (2.2.2) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$, ΔA and $v(t) \in \mathbb{R}^m$ represents parametric uncertainty and external disturbance, respectively, and $u(t) \in \mathbb{R}^m$ is the control input. When the parametric uncertainty ΔA and the disturbance $v(t)$ satisfy *matched conditions*, namely $\Delta A = BD$ and $v(t) = Bw(t)$, where $w(t)$ is now the external disturbance, (2.2.1) can be expressed in the form

$$\dot{x}(t) = Ax(t) + B(u(t) + p(t)),$$

where $p(t) = Dx(t) + w(t)$. It is well known that when the uncertainty/disturbance $p(t)$ is bounded by some known given function, the system can be controlled robustlly (see [9], [16], [23], [24], and references therein). In practice, however, it may be difficult to obtain such a function for the specific class of systems. Thus, it is required to estimate the uncertainty/disturbance without *a priori* knowledge.

In this study, it is shown that such uncertainty/disturbance can be estimated by the inverse problem of tracking and the use of a feedforward filter, with an appropriately designed input to the observer-like system. Consider the open-loop system, henceforth known as the *real system*:

$$\dot{r}(t) = Ar(t) + Bp(t), \tag{2.2.3}$$

where p(t) represents the uncertainty/disturbance. Define an 'observer-like' system as follows:

$$\dot{x}(t) = Ax(t) + B\bar{u}(t). \tag{2.2.4}$$

Note that the system matrices of the real system and observer-like system are identical. Next, consider the tracking problem such that state $x(t)$ follows $r(t)$. If there exists an appropriate control input $\bar{u}(t)$ such that tracking is almost perfectly achieved, then

$$\begin{aligned} \dot{r}(t) &= Ar(t) + Bp(t) \\ \dot{x}(t) &= Ax(t) + B\bar{u}(t) \\ r(t) &\approx x(t) \end{aligned}$$

imply that $p(t) \approx \bar{u}(t)$ for t sufficiently large. In other words, tracking $r(t)$ by $x(t)$ enables one to estimate some unknown disturbance $p(t)$.

Consider the closed-loop system described by following equations:

$$\begin{aligned} \dot{r}(t) &= Ar(t) + B(u(t) + p(t)), & (2.2.5) \\ \dot{x}(t) &= Ax(t) + B\bar{u}(t), & (2.2.6) \\ u(t) &= -\bar{u}(t). & (2.2.7) \end{aligned}$$

In this case, the estimated disturbance is fed back to the real system to cancel out the uncertainty/disturbance. It is expected that this closed-loop system is robust with respect to the uncertainty/disturbance. However, it is impossible to estimate the disturbance by tracking if the estimated disturbance is fed back to the real system. Since $p(t) \approx \bar{u}(t)$, for t sufficiently large, then, defining $\epsilon(t) := u(t) + p(t)$, (2.2.5) can be expressed as.

$$\begin{aligned} \dot{r}(t) &= Ar(t) + B(u(t) + p(t)) \\ &= Ar(t) + B\epsilon(t) \\ \epsilon(t) &\approx 0 \end{aligned}$$

Hence, the function $\epsilon(t)$ is estimated, for t sufficiently large. Thus, unlike the open-loop system describe by (2.2.3) and (2.2.4), tracking $r(t)$ does not imply estimation of the uncertainty/disturbance. To overcome this problem, a feed-forward filter and a modified reference signal are introduced. The feedforward filter is designed as follows:

$$\begin{aligned} \dot{x}_f(t) &= Ax_f(t) + Bu(t) & (2.2.8) \\ x_f(t_0) &= x_f^0, \end{aligned}$$

where x_f^0 is specified. Note that the system matrices of this filter are exactly the same as the system matrices of the real system and the input to this filter is exactly the same as the input to system (2.2.5).

Define the modified reference signal as follows:

$$\bar{r}(t) := r(t) - x_f(t) \quad (2.2.9)$$

so that

$$\dot{\bar{r}}(t) = A\bar{r}(t) + Bp(t). \quad (2.2.10)$$

Regardless of the existence of the control input to the real system (2.2.5), the input to the system (2.2.10) is always equivalent to uncertainty/disturbance to the real system. Hence, for estimation purposes, the equation

$$\dot{\bar{r}}(t) = A\bar{r}(t) + Bp(t)$$

is considered together with the observer-like system

$$\dot{x}(t) = Ax(t) + B\bar{u}(t). \quad (2.2.11)$$

This is exactly the same the set of equations constructed for the open-loop system (see (2.2.3) and (2.2.4)). Thus, tracking the state of the modified reference signal by an observer-like system enables one to estimate the uncertainty/disturbance for closed-loop system. However, the questions that need to be answered are:

- How is the controller $\bar{u}$ determined for tracking?
- Under what hypotheses can this disturbance estimation and cancellation method be used?

The answer to these questions are given in later sections.

2.3 Preliminaries

In this section, before the main result is presented, some preliminary work is provided. Firstly, the error system, which is used for analysis, is defined and then assumptions, used for analysis purposes, are given. Finally, the systems used for this method are defined.

2.3.1 Definition of the error system

In this section, the error system, which is used for analysis of the closed-loop system (2.2.5)-(2.2.9) is introduced.

Recall that the dynamics of the modified reference signal (2.2.10) and the observer-like system (2.2.6) are given as follows:

$$\begin{aligned} \dot{\bar{r}}(t) &= A\bar{r}(t) + Bv(t) \\ \dot{x}(t) &= Ax(t) + B\bar{u}(t), \end{aligned}$$

where $\bar{r}(t) \in \mathbb{R}^n$ is the modified reference signal, defined in (2.2.9), $x(t) \in \mathbb{R}^n$ is state vector of observer-like system, $v(t) \in \mathbb{R}^m$ is matched uncertainty/disturbance, and $\bar{u}(t) \in \mathbb{R}^m$ is a control input. Also, the system is assumed to be transformed into controllable canonical form (see [7] and [12] for more details).

The error system is defined as follows:

$$e(t) := x(t) - \bar{r}(t) \tag{2.3.1}$$
$$\dot{e}(t) = Ae(t) + B(\bar{u}(t) - v(t)). \tag{2.3.2}$$

Define $\bar{u}(t)$ to be the error feedback control

$$\bar{u}(t) = K(t)e(t). \tag{2.3.3}$$

Note that the feedback gain of the error system is *not* fixed but **time-varying**. Substituting (2.3.3) into (2.3.2) gives

$$\dot{e}(t) = (A + BK(t))e(t) - Bv(t). \tag{2.3.4}$$

The relation of real system, observer-like system, feedforward filter, and error-dynamics is illustrated in Figure 2.3.1 as block-diagram form.

2.3.2 Assumptions

In this subsection, basic assumptions are introduced.

Assumption 1 *(A, B) is a controllable pair.*

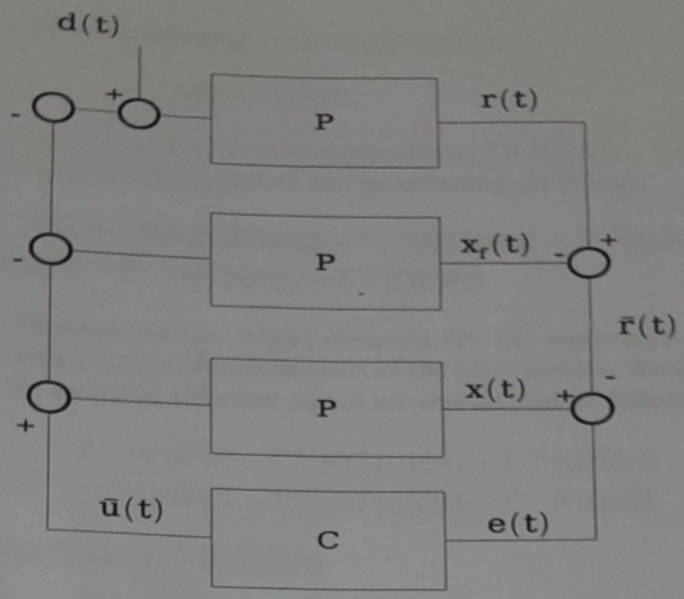

Figure 2.3.1: Block diagram of real system, observer-like system, and feedforward filter.

Remark 5 In view of Assumption 1, a linear transformation of coordinates can be applied to transform the system matrices A and B into controllable canonical form. Hence, without loss of generality, it is assumed that the system matrices A and B are in controllable canonical form.

Assumption 2 *Nominal system is stable; i.e. real parts of all eigenvalues of A are negative.*

Assumption 3 *All states of the real system are available for control purposes.*

Assumption 4 *The norm of the external disturbance/uncertainty $v(t)$ is bounded by some constant, which is assumed to be unknown, that is*

$$||v(t)|| \leq \alpha,$$

where α is some unknown constant.

Assumption 5 *The eigenvalues of $A + BK(t)$ are distinct, where definition of $K(t)$ is given at later section for each class of systems (see (2.4.2)-(2.4.3), (2.5.2)-(2.5.3), and (2.6.6)).*

2.3.3 Diagonalisation of error system

By Assumption 5, the feedbacked system matrix of the error system (2.3.4), $A + BK(t)$ can be diagonalised by a linear coordinate transformation (see [14] for details.). Consider the following linear transformation:

$$e(t) = T(t)\bar{e}(t),$$

where the column vectors of $T(t)$ are the eigenvectors of $A + BK(t)$. Using this transformation, the error system (2.3.4) can be expressed as follows:

$$\begin{aligned}\dot{\bar{e}}(t) &= T^{-1}(t)(A + BK(t))T(t)\bar{e}(t) - T^{-1}(t)\dot{T}(t)\bar{e}(t) - T^{-1}(t)Bv(t) \\ &= \Lambda(t)\bar{e}(t) - T^{-1}(t)\dot{T}(t)\bar{e}(t) - T^{-1}(t)Bv(t), \end{aligned} \tag{2.3.5}$$

where $\Lambda(t)$ is a diagonal matrix, whose elements are the eigenvalues of $A + BK(t)$. Corresponding to this transformation of the error system, the observer-like system and the modified reference signal are represented as follows:

$$\dot{\bar{x}}(t) = (T^{-1}(t)AT(t) - T^{-1}(t)\dot{T}(t))\bar{x}(t) + T^{-1}(t)B\bar{u}(t) \tag{2.3.6}$$

$$\dot{\tilde{r}}(t) = (T^{-1}(t)AT(t) - T^{-1}(t)\dot{T}(t))\tilde{r}(t) + T^{-1}(t)Bv(t) \tag{2.3.7}$$

where $x(t) = T(t)\bar{x}(t)$ and $\bar{r}(t) = T(t)\tilde{r}(t)$.

Remark 6 If pole assignment is to be considered, this can be incorporated in the design procedure. For example, if a constant matrix F is designed then, the real system takes the form

$$\begin{aligned}\dot{r}(t) &= (A + BF)r(t) + B(p(t) + u(t)) \\ &= \bar{A}r(t) + B(p(t) + u(t)).\end{aligned}$$

Then, defining an observer-like system and a feedforward filter with respect to $\bar{A}$, the same relation as above is obtained. Therefore, the problem of robust pole assignment can be treated using the same formulation.

Remark 7 $T(t)$ matrices, which are used for coordinate transformations, are defined at later sections for each classes of systems.

Remark 8 In [11], the analysis is performed by transforming the original system using a time-varying coordinate transformation. In this study, as described as above, the matrix $A + BK(t)$ is transformed into diagonal matrix by a time-varying transformation. In terms of coordinate transformations, the idea of analyzing the transformed system is the same. However, both methods consider a different problem and use different adaptive algorithms.

Remark 9 Recall that the error $e(t)$ is

$$\begin{aligned}e(t) &= x(t) - \bar{r}(t) \\ &= x(t) - (r(t) - x_f(t)) \\ &= (x(t) + x_f(t)) - r(t).\end{aligned}$$

Consider the dynamics of $x(t) + x_f(t)$. Now

$$\begin{aligned}\dot{x}(t) + \dot{x}_f(t) &= (Ax(t) + B\bar{u}(t)) + (Ax_f(t) + Bu(t)) \\ &= A(x(t) + x_f(t)) + B(u(t) + \bar{u}(t)).\end{aligned} \tag{2.3.8}$$

Thus, treating $x(t) + x_f(t)$ as a new state variable, the formulation (2.3.8) is equivalent to an observer-like system, which is used in disturbance estimation methods of [36], [35], and [38].

Although, both representations are identical, for this study the representation with an observer-like system and feedforward filter, defined previously, is used even though it is not standard. The reasons for this are as follows.

1. The original motivation of this study comes from the idea 'tracking a real system by an observer-like system with the same system matrix should result in the estimation of the uncertainty/disturbance'.

2. Using the observer-like system with the feedforward filter representation, it can be clearly understood that the observer-like system tracks the real system.

3. If a feedforward filter is not utilised, then the observer-like system will give rise to the system $\dot{x}(t) = Ax(t)$, which is independent of any input. Thus, it is not obvious that $x(t)$ will track $r(t)$.

Therefore, although the technique is not standard, an observer-like system with feedforward filter representation is used for this study.

2.4 Second order single-input system

In this section, an adaptive algorithm, theorems, and simulation example for the estimation and cancellation of uncertainty/disturbance are presented. Firstly, an adaptive algorithm is described. Next, associated lemmas are constructed which are required to prove the main result. Then the main theorem is given, and, finally, a simulation example is included, which shows the performance of the method proposed.

2.4.1 Adaptive algorithm

In this subsection, an adaptive algorithm is presented. The main idea of this adaptive algorithm is that by decreasing the (real, negative) eigenvalues of $A+BK(t)$, the norm of $\bar{e}(t)$, which is the transformed error state, will converge to within any specified interval $(0, \epsilon_e)$ in the presence of bounded disturbances. Note that as a result of the adaptive law, the error system is a time-varying system. The adaptive algorithm is based on evaluation of a Lyapunov-like function. In general, for a time-varying system, constructing a candidate Lyapunov function can be difficult. However, it is shown that in the transformed coordinates, it is always possible to construct a Lyapunov-like function and, hence, the adaptive algorithm is realizable. Using this adaptive algorithm and Lyapunov-like function, the norm of the transformed error state converges to a prescribed set, and, hence, the uncertainty/disturbance can be estimated. Initially, a second order single-input system is considered.

Algorithm 1 *The eigenvalues of $A+BK(t)$, $\lambda_i(t)$, are determined as follows. Suppose ϵ_e, δ, and κ_2 are prescribed positive constants, which are determined by control designer. Define $V(t) := ||\bar{e}(t)||^2$.*

One eigenvalue, say $\lambda_1(t)$, is determined as follows. At $t = t_0$, $\lambda_1(t) := -\lambda_0$ and $\dot{\lambda}_1(t) := -\lambda_{d0}$, where $\lambda_0 \in \mathbb{R}^+$ and $\lambda_{d0} := 0$ or δ are chosen by the control designer. The structure of $\dot{\lambda}_1(t)$ is determined as follows:

1. *let $\tau := t$ $(t \geq t_0)$;*
2. *evaluate $\bar{e}(\tau)$ and, hence, $V(\tau)$ is also obtained;*
3. (a) *if $(V(\tau) \leq \epsilon_e^2$ and $\dot{\lambda}_1(\tau) = -\delta)$ then $\tau_1 = \tau$ and $\dot{\lambda}_1(\cdot)$ has the following structure: $\dot{\lambda}_1(s) = f(s, \tau_1)$ for $s \geq \tau_1$;*
   (b) *or if $(V(\tau) > \epsilon_e^2$ and $\dot{\lambda}_1(\tau) = 0)$ then $\tau_1 = \tau$ and $\dot{\lambda}_1(\cdot)$ has the structure: $\dot{\lambda}_1(s) = g(s, \tau_1)$ for $s \geq \tau_1$;*
   (c) *otherwise, the structure of $\dot{\lambda}_1(\cdot)$ is not changed;*
4. *$t = t + \Delta t$ where Δt is a prescribed positive constant;*
5. *evaluate $\dot{\lambda}_1(t)$ using the given structure of $\dot{\lambda}_1(s)$;*

where $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are continuous functions defined later.

Then $\lambda_2(t)$ is determined as follows:

$$\lambda_2(t) = \kappa_2 \lambda_1(t), \tag{2.4.1}$$

where $\kappa_2 > 0$ and $\kappa_2 \neq 1$.

Remark 10 At step 2 in the algorithm, $\bar{e}(\tau)$ can be approximately determined by using a numerical method to obtain an approximate solution of the ordinary differential equation (see [31]), for example when values at t is known, and if values at $t + \Delta t$ is required to be obtained, using euler method,

$$x(t + \Delta t) \approx x(t) + \dot{x}(t)\Delta t$$

where Δt is prescribed positive constant and $\dot{x}(t)$ is determined by (2.2.11) and (2.3.3). Simirary, $x_f(t + \Delta t)$ is determined using (2.2.8), (2.2.7), (2.3.3), and euler method or alike. Then, $\bar{r}(t + \Delta t)$ is determined by (2.2.9). Also, $e(t + \Delta t)$ is determined by (2.3.1). Finally, applying time-varying linear coordinate transformation, $\bar{e}(t + \Delta t)$ is determined; i.e. $\bar{e}(t + \Delta t) = T(t + \Delta t)^{-1} e(t + \Delta t)$.

Remark 11 As described earlier, the basic idea of this adaptive algorithm is to decrease the eigenvalues of $A + BK(t)$, which is a feedbacked system matrix of error system, so that the norm of the transformed error state is small enough. If this norm is small enough, the dynamics of the observer-like system and the real system will be almost the same for t sufficiently large and, hence, disturbance can be estimated using the control input to the observer-like system.

To implement this idea, in this adaptive algorithm, the following criteria are used.

1. If the Lyapunov-like function has a value which is smaller than the given constant, then the eigenvalues are no longer decreased but are kept at some constant value, i.e. $\dot{\lambda}_1(t) = 0$.

2. If the value of the Lyapunov-like function is greater than the given constant, then the eigenvalues are decreased, i.e. $\dot{\lambda}_1(t) = -\delta$.

However, if this idea is implemented directly, $\dot{\lambda}$ will be discontinuous. Thus, in the algorithm, this has been accounted for by the introduction of the continuous functions f and g. Particular examples of such functions are

$$f(t,\tau) := \begin{cases} \frac{1}{2}\delta\sin(\omega(t-\tau)+\frac{3}{2}\pi)-\frac{1}{2}\delta, & \tau \le t \le \frac{\pi}{\omega}+\tau \\ 0, & \frac{\pi}{\omega}+\tau < t \end{cases},$$

$$g(t,\tau) := \begin{cases} \frac{1}{2}\delta\sin(\omega(t-\tau)+\frac{1}{2}\pi)-\frac{1}{2}\delta, & \tau \le t \le \frac{\pi}{\omega}+\tau \\ -\delta, & \frac{\pi}{\omega}+\tau < t \end{cases},$$

where ω is a prescribed positive constant.

Remark 12 The second order system, defined by the pair (A,B), is assumed to be controllable (see Assumption 1). Hence, a linear transformation of coordinates can be applied to transform the system into controllable canonical form [12], that is

$$A = \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Remark 13 When the eigenvalues of $A+BK(t)$ are determined, the feedback gain of the observer-like system, $K(t) = [k_1(t)\ k_2(t)]$, is determined as follows:

$$k_1(t) = -(\lambda_1(t)+\lambda_2(t)) - a_{21} \quad (2.4.2)$$

$$k_2(t) = -\lambda_1(t)\lambda_2(t) - a_{22} \quad (2.4.3)$$

where a_{ij} is the element in the i^{th} row and j^{th} column of the matrix A in controllable canonical form.

Remark 14 For a second order system, a $T(t)$ matrix can be expressed as follows:

$$T(t) = \begin{bmatrix} 1 & 1 \\ \lambda_1(t) & \lambda_2(t) \end{bmatrix}. \quad (2.4.4)$$

The matrix $T(t)$ is known as the Vandermonde matrix ([30], [33], and [34]).

The work on control gain adaptation, although a related field, does not imply estimation and cancellation of uncertainty to the system. In this case, the gain is determined adaptively so that the system with uncertainty/disturbance or system, whose system matrix parameters are unknown, is stabilized or tracks the given reference signal. In general, there are two classes of adaptive law in the literature. For the first one, the gain is determined by a differential equation (see [1], [11], and [15], for example). For second one, the gain is determined by switching between a set of gains (see [6], for example).

2.4.2 Main Result

Some characteristics of the matrix $T(t)$ are given in the following two lemmas. Firstly, it is shown that $||T^{-1}(t)||_1$ is a non-increasing function. Then, it is shown that $||\dot{T}(t)||$ is bounded by a constant for all time t.

Lemma 1 *If Algorithm 1 is used, $||T^{-1}(t)||_1$ is a non-increasing function.*

Proof In view of (2.4.4),

$$T^{-1}(t) = \frac{1}{\lambda_2(t) - \lambda_1(t)} \begin{bmatrix} \lambda_2(t) & -1 \\ \lambda_1(t) & 1 \end{bmatrix}$$

Hence, the 1-norm of $T^{-1}(t)$ is given by,

$$\begin{aligned} ||T^{-1}(t)||_1 &= \frac{1}{|\lambda_1(t) - \lambda_2(t)|}(|\lambda_2(t)| + |\lambda_1(t)| + 2) \\ &= \frac{1}{|\lambda_1(t) - \lambda_2(t)|}(-\lambda_2(t) - \lambda_1(t) + 2) \end{aligned}$$

By Algorithm 1, $\lambda_2(t) = \kappa_2\lambda_2(t)$ and, hence,

$$\begin{aligned} ||T^{-1}(t)||_1 &= \frac{1}{|\lambda_1(t) - \kappa_2\lambda_1(t)|}(-\kappa_2\lambda_1(t) - \lambda_1(t) + 2) \\ &= \frac{1+\kappa_2}{|1-\kappa_2|} + \frac{2}{|1-\kappa_2||\lambda_1(t)|} \end{aligned} \tag{2.4.5}$$

Since, $t \mapsto |\lambda_1(t)|$ is a non-decreasing function, (2.4.5) implies that $||T^{-1}(t)||_1$ is a non-increasing function. ■

Lemma 2 *If Algorithm 1 is used, $||\dot{T}(\cdot)||$ is bounded.*

Proof For a second order single-input single-output system, in controllable canonical form, the transformation has the form

$$T(t) = \begin{bmatrix} 1 & 1 \\ \lambda_1(t) & \lambda_2(t) \end{bmatrix},$$

where $\lambda_i(t)$ are the eigenvalues of $A + BK(t)$ and $K(t)$ is defined by (2.4.2) and (2.4.3). Hence, the time derivative of $T(t)$ is given by

$$\dot{T}(t) = \begin{bmatrix} 0 & 0 \\ \dot{\lambda}_1(t) & \dot{\lambda}_2(t) \end{bmatrix},$$

and so

$$\dot{T}^t(t)\dot{T}(t) = \begin{bmatrix} \dot{\lambda}_1^2(t) & \dot{\lambda}_1(t)\dot{\lambda}_2(t) \\ \dot{\lambda}_1(t)\dot{\lambda}_2(t) & \dot{\lambda}_2^2(t) \end{bmatrix}.$$

The characteristic equation of $\dot{T}^t(t)\dot{T}(t)$ is

$$\begin{aligned} \left|\dot{T}^t(t)\dot{T}(t) - \sigma I\right| &= \begin{vmatrix} \dot{\lambda}_1^2(t) - \sigma & \dot{\lambda}_1(t)\dot{\lambda}_2(t) \\ \dot{\lambda}_1(t)\dot{\lambda}_2(t) & \dot{\lambda}_2^2(t) - \sigma \end{vmatrix} \\ &= (\dot{\lambda}_1^2(t) - \sigma)(\dot{\lambda}_2^2(t) - \sigma) - (\dot{\lambda}_1(t)\dot{\lambda}_2(t))^2 \\ &= \sigma\left(\sigma - (\dot{\lambda}_1(t) + \dot{\lambda}_2(t))\right) \\ &= 0 \end{aligned} \tag{2.4.6}$$

Hence, the time-varying eigenvalues of $\dot{T}^t(t)\dot{T}(t)$ are specified by $\left\{0, \dot{\lambda}_1^2(t) + \dot{\lambda}_2^2(t)\right\}$. Therefore, using (2.4.1),

$$\begin{aligned} ||\dot{T}(t)|| &= \sqrt{\sigma_{\max}(\dot{T}^t(t)\dot{T}(t))} \\ &= \sqrt{\dot{\lambda}_1^2(t) + \dot{\lambda}_2^2(t)} \\ &= |\dot{\lambda}_1(t)|\sqrt{1+\kappa_2^2}. \end{aligned} \tag{2.4.7}$$

Since $|\dot{\lambda}_1(t)|$ is bounded, then, from (2.4.7), it follows that $||\dot{T}(\cdot)||$ is bounded. ■

In the following two lemmas, it is shown that, provided a certain condition holds and the norm of the transformed error state is greater than a specified function, $\dot{V}(t) < 0$ holds and, in addition, the specified function is decreasing.

Lemma 3 *Under the dynamics of (2.3.4), if $\sigma_{\max}(\Lambda(t)) + \xi||T^{-1}(t)||_1 < 0$ and if $||\bar{e}(t)|| > \epsilon(t)$, with*

$$\epsilon(t) = -\frac{\alpha||T^{-1}(t)||_1||B||}{\sigma_{\max}(\Lambda(t)) + \xi||T^{-1}(t)||_1}, \tag{2.4.8}$$

holds for all t in some interval $[\tau_1, \tau_2]$ with $\tau_2 > \tau_1 \geq t_0$, where ξ is a positive constant which satisfies $||\dot{T}(t)|| \leq \xi$ for all t and $\lambda_i(t)$ satisfy the conditions specified in Algorithm 1, then for almost all $t \in [\tau_1, \tau_2]$, $\dot{V}(t) < 0$ along solutions to (2.3.5).

Proof The time derivative of function $V(t)$ along solutions to (2.3.5) satisfies

$$\begin{aligned} \dot{V}(t) &= \dot{\bar{e}}^t(t)\bar{e}(t) + \bar{e}^t(t)\dot{\bar{e}}(t) \\ &\leq 2(\sigma_{\max}(\Lambda(t)) + ||\dot{T}(t)||||T^{-1}(t)||_1)||\bar{e}(t)||^2 + 2\alpha||T^{-1}(t)||_1||B||\,||\bar{e}(t)||. \end{aligned}$$

By Lemma 2, there exists positive constant ξ such that $||\dot{T}(t)|| \leq \xi$ holds for all t. Hence,

$$\dot{V}(t) \leq 2(\sigma_{\max}(\Lambda(t)) + \xi||T^{-1}(t)||_1)||\bar{e}(t)||^2 + 2\alpha||T^{-1}(t)||_1||B||\,||\bar{e}(t)|| \tag{2.4.9}$$

Since $||\bar{e}(t)|| > \epsilon(t)$, then it can be easily shown that

$$2(\sigma_{\max}(\Lambda(t)) + \xi||T^{-1}(t)||_1)||\bar{e}(t)||^2 + 2\alpha||T^{-1}(t)||_1||B||\,||\bar{e}(t)|| < 0.$$

Hence, if $||\bar{e}(t)|| > \epsilon(t)$ and $\sigma_{\max}(\Lambda(t)) + \xi||T^{-1}(t)||_1 < 0$ holds for all $t \in [\tau_1, \tau_2]$, the result follows. ■

Lemma 4 *Under the dynamics of (2.3.4), if $\lambda_i(t)$ satisfy the conditions given in Algorithm 1, $||\bar{e}(t)|| > \epsilon_e$, where ϵ_e is prescribed constant, and $\sigma_{\max}(\Lambda(t)) + \xi||T^{-1}(t)||_1 < 0$ hold, where ξ is a positive constant such that $||\dot{T}(t)|| \leq \xi$ holds for all t, then $\epsilon(\cdot)$ is a decreasing function.*

Proof In view of Lemma 1 and since, $\bar{e}(t) \notin \mathbb{B}(\epsilon_e)$, $||T^{-1}(t)||_1$ is a decreasing function. Also, since $\bar{e}(t) \notin \mathbb{B}(\epsilon_e)$, $\sigma_{\max}(\Lambda(t))$ is a decreasing function when Algorithm 1 is used. Thus, the numerator of the expression defining $\epsilon(t)$ is a decreasing function and the denominator of expression defining $\epsilon(t)$ is decreasing function. Therefore, if Algorithm 1 is used, and if $||\bar{e}(t)|| > \epsilon_e$ and if $\sigma_{\max}(\Lambda(t)) + \xi||T^{-1}(t)||_1 < 0$ holds, $\epsilon(\cdot)$ is a decreasing function. ■

At following two lemmas, analogue to previous lemmas, it is shown that, provided a certain condition holds and the norm of the transformed error state is greater than a specified function, $\dot{V}(t) < -\mu$ holds and, in addition, the specified function is decreasing.

Lemma 5 *Suppose μ is any positive constant. If, for all t in some interval $[\tau_1, \tau_2]$ with $\tau_2 > \tau_1 \geq t_0$, $||\bar{e}(t)|| > \bar{\epsilon}(t)$ under the dynamics of system (2.3.4), where*

$$\bar{\epsilon}(t) = \frac{\gamma(t) + \sqrt{\gamma^2(t) - 4\beta(t)\mu}}{-2\beta(t)} \tag{2.4.10}$$

and

$$\beta(t) := 2(\sigma_{\max}(\Lambda(t)) + \xi||T^{-1}(t)||_1), \tag{2.4.11}$$

$$\gamma(t) := 2\alpha||T^{-1}(t)||_1||B||, \tag{2.4.12}$$

and $\sigma_{\max}(\Lambda(t)) + \xi||T^{-1}(t)||_1 < 0$ holds, where ξ is a positive constant which satisfies $||\dot{T}(t)|| \leq \xi$ for all t, then $\dot{V}(t) < -\mu$ along solutions to (2.3.5).

Proof Recall, from (2.4.9), along solutions to (2.3.5).

$$\dot{V}(t) \leq \beta(t)||\bar{e}(t)||^2 + \gamma(t)||\bar{e}(t)||$$

Since $||\bar{e}(t)|| > \bar{\epsilon}(t)$, where $\bar{\epsilon}(t)$ is defined in (2.4.10),

$$\beta(t)||\bar{e}(t)||^2 + \gamma(t)||\bar{e}(t)|| \leq -\mu$$

and the result follows. ■

Remark 15 It is clear that, by definition of $\epsilon(t)$ and $\bar{\epsilon}(t)$, for any given $\mu > 0$, $\epsilon(t) < \bar{\epsilon}(t)$ for all t.

Lemma 6 *Under the dynamics of (2.3.4), if $\lambda_i(t)$ satisfy the conditions specified in Algorithm 1 and $||\bar{e}(t)|| > \epsilon_e$ for $t \in [\tau_1, \tau_2]$, where $\tau_2 > \tau_1 > t_0$, hold, then for any $\mu > 0$, $\bar{\epsilon}(\cdot)$ is a decreasing function in $[\tau_1, \tau_2]$ where $\tau_1 \geq t_1 > t_0$ and t_1 is a finite time for which $\sigma_{\max}(\Lambda(t_1)) + \xi||T^{-1}(t_1)||_1 < 0$ holds, where ξ is a positive constant which satisfies $||\dot{T}(t)|| \leq \xi$ for all t.*

Proof Consider $\beta(t)$ and $\gamma(t)$ defined in (2.4.11) and (2.4.12). Since $\bar{e}(t) \notin \mathbb{B}(\epsilon_e)$ for all $t \in [\tau_1, \tau_2]$, $\gamma(t)$ is a positive decreasing function and $\beta(t)$ is a negative decreasing function. Thus $\dot{\gamma}(t) \leq 0$ and $\dot{\beta}(t) < 0$ $\forall t \in [\tau_1, \tau_2]$. Using the above information, the time derivative of $\bar{\epsilon}(t)$, $\forall t \in [\tau_1, \tau_2]$, is given by

$$\begin{aligned}
\frac{d\bar{\epsilon}(t)}{dt} &= -\frac{1}{2\beta(t)}\dot{\gamma}(t) + \frac{1}{2\beta^2(t)}\gamma(t)\dot{\beta}(t) + \frac{1}{2\beta^2(t)}\dot{\beta}(t)\sqrt{\gamma^2(t) - 4\beta(t)\mu} \\
&\quad - \frac{1}{4\beta(t)}\frac{1}{\sqrt{\gamma^2(t) - 4\beta(t)\mu}}(2\gamma(t)\dot{\gamma}(t) - 4\dot{\beta}(t)\mu) \\
&\leq \frac{1}{2\beta^2(t)}\dot{\beta}(t)\sqrt{\gamma^2(t) - 4\beta(t)\mu} - \frac{1}{4\beta(t)}\frac{1}{\sqrt{\gamma^2(t) - 4\beta(t)\mu}}(-4\dot{\beta}(t)\mu) \\
&= \frac{\dot{\beta}(t)(\gamma^2(t) - 2\beta(t)\mu)}{2\beta^2(t)\sqrt{\gamma^2(t) - 4\beta(t)\mu}} < 0,
\end{aligned}$$

for all $t \in [\tau_1, \tau_2]$ and $\mu > 0$. Then, the result follows. ■

In following lemma, it is shown that it is always possible for a trajectory of the error system to reach some prescribed set by finite time. This lemma provides a foundation for reachability of the prescribed set by designing a suitable control input.

Lemma 7 *Under the dynamics of system (2.3.4), if $\lambda_i(t)$ satisfy the conditions specified in Algorithm 1, then there exists $t^* > t_1$ such that $\bar{e}(t) \in \mathbb{B}(\epsilon(t_1))$ for $t > t^*$, where t_1 is any finite time such that*

$$\sigma_{\max}(\Lambda(t_1)) + \xi \|T^{-1}(t_1)\|_1 < 0 \tag{2.4.13}$$

holds, where ξ is a positive constant which satisfies $\|\dot{T}(t)\| \leq \xi$ for all t.

Proof Suppose $\|\bar{e}(t)\| > \epsilon_e$ for $t_2 \geq t \geq t_1$, then by Lemma 4, $\epsilon(t)$ is a decreasing function. Thus,

$$\epsilon(t_1) - \epsilon(t_2) = \kappa > 0,$$

where $t_2 > t_1$. By definition,

$$\bar{\epsilon}(t_2) - \epsilon(t_2) = \frac{-\gamma(t_2) + \sqrt{\gamma^2(t_2) - 4\beta(t_2)\mu}}{-2\beta(t_2)}. \tag{2.4.14}$$

Suppose the following inequality is satisfied:

$$\mu < \frac{(\gamma(t_2) - 2\beta(t_2)\kappa)^2 - \gamma^2(t_2)}{-4\beta(t_2)}.$$

This inequality gives

$$\gamma^2(t_2) - 4\beta(t_2)\mu < (\gamma(t_2) - 2\beta(t_2)\kappa)^2$$

and so, using (2.4.14),

$$\kappa > \frac{-\gamma(t_2) + \sqrt{\gamma^2(t_2) - 4\beta(t_2)\mu}}{-2\beta(t_2)} = \bar{\epsilon}(t_2) - \epsilon(t_2).$$

Hence,

$$\epsilon(t_1) - \epsilon(t_2) = \kappa > \bar{\epsilon}(t_2) - \epsilon(t_2),$$

and so

$$\bar{\epsilon}(t_2) < \epsilon(t_1). \tag{2.4.15}$$

Suppose $\|\bar{e}(t_2)\| > \epsilon(t_1) \geq \epsilon_e$ is satisfied (otherwise, the objective of this lemma is satisfied). Thus, in view of Lemma 6 and (2.4.15),

$$\epsilon(t_1) > \bar{\epsilon}(t_2) \geq \bar{\epsilon}(t), \quad t \geq t_2. \tag{2.4.16}$$

Suppose, for $t \in [t_2, \tau_2]$, $\|\bar{e}(t)\| > \bar{\epsilon}(t_2)$. Otherwise, since $\bar{\epsilon}(t_2) < \epsilon(t_1)$, $\|\bar{e}(t)\| < \epsilon(t_1)$ and so there is nothing to prove. If $\|\bar{e}(t)\| > \bar{\epsilon}(t_2)$ then, by (2.4.16), it follows that $\|\bar{e}(t)\| > \bar{\epsilon}(t)$ for $t \in [t_2, \tau_2]$. Therefore, by Lemma 5,

$$\dot{V}(t) < -\mu, \quad t \in [t_2, \tau_2],$$

along solutions to (2.3.5). Hence for $t \in [t_2, \tau_2]$,

$$\begin{aligned} ||\bar{e}(t)||^2 &= ||\bar{e}(t_2)||^2 + \int_{t_2}^{t} \dot{V}(t)\mathrm{d}t \\ &< ||\bar{e}(t_2)||^2 + \int_{t_2}^{t} -\mu \mathrm{d}t \\ &= ||\bar{e}(t_2)||^2 - \mu(t - t_2). \end{aligned} \tag{2.4.17}$$

Note that, since $||\bar{e}(t_2)|| > \epsilon(t_1)$, $||\bar{e}(t_2)||^2 - \epsilon^2(t) > 0$ and so

$$t^* := t_2 + \frac{||\bar{e}(t_2)||^2 - \epsilon^2(t_1)}{\mu}$$

satisfies $t^* > t_2$. Thus, for $t > t^*$, it follows from (2.4.17) that

$$\begin{aligned} ||\bar{e}(t)||^2 &= ||\bar{e}(t_2)||^2 - \mu(t - t_2) \\ &< ||\bar{e}(t_2)||^2 - \mu(t^* - t_2) \\ &= \epsilon^2(t_1) \end{aligned}$$

Moreover, since $\mu > 0$, t^* is finite. ■

Remark 16 Note that t_1, of Lemma 7, is any value such that inequality (2.4.13) holds.

The following lemmas and theorem show that, using Algorithm 1, it is possible to estimate and cancel the disturbance in system (2.2.5). Firstly, it is shown that a trajectory of the error system (2.3.5) always reaches a prescribed set utilising finite eigenvalues of the error system. The prescribed set is defined in terms of a parameter which is available to the control designer for tuning purposes. By incorporating this parameter in the control design, it is possible to make the estimation error for the disturbance as small as possible. Following this, it is shown that the norm of the transformed error state is uniformly bounded. Finally, it is shown that estimation and cancellation of the disturbance can be achieved under assumptions that are specified.

Note that $\epsilon(t)$, which is defined by (2.4.8), is a non-increasing function and, therefore, converges to a non-negative constant, say there exists ν such that $\lim_{t\to\infty} \epsilon(t) = \nu$. In fact, it can be shown that $\nu \neq 0$ and this is shown in the following lemma.

Lemma 8 *If $\lambda_i(t)$ satisfy the conditions given in Algorithm 1, and if $\sigma_{\max}(\Lambda(t_1)) + \xi||T^{-1}(t_1)||_1 < 0$ ($t_1 > t_0$), where ξ is a positive constant such that $||\dot{T}(t)|| \leq \xi$ holds for all t, then, under the dynamics of (2.3.4), there exists a positive constant ν such that $\lim_{t\to\infty} \epsilon(t) = \nu$, where $\epsilon(t)$ is defined in (2.4.8).*

Proof A contradiction argument is utilized to show the statement of the Lemma. Assume $\lim_{t\to\infty} \epsilon(t) = 0$ holds. Then, there exists $t_2 \geq t_1$ such that $\epsilon(t_2) = \epsilon_e$ holds. Hence, by Lemma 7, there exists $t_3 > t_2$ such that $||\bar{e}(t)|| \leq \epsilon_e$ for $t > t_3$. Also, since $||\bar{e}(t)|| \leq \epsilon_e$ for $t > t_3$, by Algorithm 1, there exists $t_4 > t_3$ such that $\dot{\lambda}_i(t) = 0$ for $t > t_4$ with $i = 1, 2$. This contradicts original assumption. Therefore, there exists positive constant ν such that $\lim_{t\to\infty} \epsilon(t) = \nu$ holds. ■

Lemma 9 *If $\lambda_i(t)$ satisfy the conditions given in Algorithm 1, and if $\sigma_{\max}(\Lambda(t_1)) + \xi||T^{-1}(t_1)||_1 < 0$ $(t_1 > t_0)$, where ξ is a positive constant which satisfies $||T(t)|| \leq \xi$ for all t, then for any prescribed $\epsilon_e > 0$, there exists $t_2 \geq t_1$ such that $e(t) \in \mathbb{B}(\epsilon_e)$ for $t \geq t_2$ and $\lambda_i(t)$ are finite for all $t \geq t_2$.*

Proof Here a contradiction argument is utilised. Assume that there does not exist t_2 such that $\dot{\lambda}_1(t) = 0$ for all $t \geq t_2$. There are two possibilities.

(i) There exist t_3, t_4, and t_5 such that following sequence will occur: $\dot{\lambda}_1(t_3) = 0$, $\dot{\lambda}_1(t_4) = -\delta$, $\dot{\lambda}_1(t_5) = 0$, an infinite number of times.

(ii) There exists t' such that $\dot{\lambda}_1(t) = -\delta$ for all $t > t'$.

Consider case (i). As a consequence of Algorithm 1, by integrating from t_3 to t_5, the following result is obtained:

$$\lambda_1(t_5) \leq \lambda_1(t_3) - \frac{1}{2}\frac{\pi}{\omega}\delta$$

Hence, if this sequence is repeated an infinite number of times,

$$\lim_{t \to \infty} \lambda_1(t) = -\infty$$

Now consider case (ii). This case is much easier. By integration,

$$\lim_{t \to \infty} \lambda_1(t) = -\infty$$

By definition of $\epsilon(t)$ in (2.4.8), both two cases imply that $\epsilon(t)$ converges to zero. However, this contradicts Lemma 8. Thus, the original assumption is invalid and so there exists t_2 such that $\dot{\lambda}_1(t) = 0$ for all $t \geq t_2$. Hence, by Algorithm 1, if $\dot{\lambda}_1(t) = 0$ for $t \geq t_2$, then $||\bar{e}(t)|| \leq \epsilon_e$ for $t \geq t_2$. ■

Lemma 10 *If algorithm 1 is used, $||\bar{e}(t)||$ is uniformly bounded under the dynamics of (2.4.8).*

Proof Recall that the time derivative of the function $V(t)$ along solutions to (2.3.5) satisfies

$$\dot{V}(t) \leq \beta(t)||\bar{e}(t)||^2 + \gamma(t)||\bar{e}(t)||$$

where $\beta(t) = \sigma_{\max}(\Lambda(t)) + \xi||T^{-1}(t)||_1$, $\gamma(t) = 2\alpha||T^{-1}(t)||_1||B||$, and ξ is a positive constant such that $||T(t)|| \leq \xi$ holds for all t. Suppose $\bar{e}(t) \notin B(\epsilon_e)$. Consider the case when $\beta(t) > 0$. Since $\lambda(t)$ is a decreasing function, $\sigma_{\max}(\lambda(t))$ is a decreasing function and, by Lemma 1, $||T^{-1}(t)||_1$ is a decreasing function. Hence, $\beta(t)$ is a decreasing function and there exists $t^* > t_0$ such that $\beta(t) < 0$ for $t > t^*$.

Since $||T^{-1}(t)||_1$ is a decreasing function, $\gamma(t)$ is a decreasing function. Therefore, $\beta(t) \leq \beta(t_0)$ and $\gamma(t) \leq \gamma(t_0)$ for $t \geq t_0$. Hence,

$$\begin{aligned} \dot{V}(t) &\leq \beta(t)||\bar{e}(t)||^2 + \gamma(t)||\bar{e}(t)|| \\ &\leq \bar{\beta}||\bar{e}(t)||^2 + \bar{\gamma}||\bar{e}(t)|| \end{aligned}$$

where $\bar{\beta} = \beta(t_0) > 0$ and $\bar{\gamma} = \gamma(t_0) > 0$.

There are two cases to consider. Suppose $||\bar{e}(t)|| < 1$ for all $t \leq t^*$. then, clearly, $||\bar{e}(t)||$ is bounded for $t \in (t_0, t^*]$.

On the other hand, suppose there exists $\bar{t} < t^*$ such that $||\bar{e}(t)|| \geq 1$ for $t^* \geq t \geq \bar{t}$. In this case, for $t \in [\bar{t}, t^*]$,

$$\begin{aligned}
\dot{V}(t) &\leq \bar{\beta}||\bar{e}(t)||^2 + \bar{\gamma}||\bar{e}(t)|| \\
&\leq \bar{\beta}||\bar{e}(t)||^2 + \bar{\gamma}||\bar{e}(t)||^2 \\
&= (\bar{\beta} + \bar{\gamma})||\bar{e}(t)||^2 \\
&= (\bar{\beta} + \bar{\gamma})V(t)
\end{aligned}$$

where $\bar{\beta} + \bar{\gamma} > 0$. Therefore, for $\bar{t} \leq t \leq t^*$,

$$V(t) \leq C\exp((\bar{\beta} + \bar{\gamma})t)$$

where C is constant, and so,

$$V(t) \leq C\exp((\bar{\beta} + \bar{\gamma})t^*),$$

that is $||\bar{e}(t)||$ is bounded for all $t \in [\bar{t}, t^*]$. Hence, $||\bar{e}(t)||$ is bounded for $t \in [t_0, t^*]$.

Next, consider $t > t^*$, in which case, $\beta(t) = \sigma_{\max}(\Lambda(t)) + \xi||T^{-1}(t)||_1 < 0$. Suppose $\epsilon(t) \geq ||\bar{e}(t)|| > \epsilon_e$ for all $t > t^*$. For $t > t^*$, $\beta(t) < 0$ and, hence, by Lemma 4, $\epsilon(t)$ is a decreasing function. Therefore,

$$||\bar{e}(t)|| \leq \epsilon(t^*), \quad \forall t > t^* \tag{2.4.18}$$

Now suppose, there exists $t^{'} > t^*$ such that

$$||\bar{e}(t^{'})|| > \epsilon(t^{'}) > \epsilon_e.$$

By Lemma 4, $\epsilon(t)$ is a decreasing function. Hence, for $t > t^{'}$,

$$||\bar{e}(t^{'})|| > \epsilon(t^{'}) > \epsilon(t).$$

Thus, in view of Lemma 3,

$$||\bar{e}(t^{'})|| > ||\bar{e}(t)||, \quad \forall t > t^{'}. \tag{2.4.19}$$

Inequalities (2.4.18) and (2.4.19) imply that $||\bar{e}(t)||$ is bounded for all $t > t^*$. Since $||\bar{e}(t)||$ is bounded for all $t \in [t_0, t^*]$ and $t \in (t^*, \infty)$, the statement of this Lemma is valid. ■

Theorem 3 *For the matched disturbance/uncertainty in system (2.2.5), say $v(t)$, and the control input to the observer-like system (2.2.4), say $\bar{u}(t)$, $\bar{u}(t) \approx v(t)$ can be achieved for t sufficiently large using the observer-like system (2.2.4), feedforward filter (2.2.8), modified reference signal (2.2.9), and feedback control (2.3.3), when the feedback gain matrix is determined using (2.4.2)-(2.4.3) and Algorithm 1.*

Proof By Lemma 9, all trajectories of (2.3.5) reach $\mathbb{B}(\epsilon_e)$ using Algorithm 1. Also by Lemmas 9 and 10, all internal signals consisting of transformed state of the error system, the eigenvalues of error system, and the feedback gain for the observer-like system, are uniformly bounded. Therefore, $||\bar{e}(t)|| \leq \epsilon_e$, for t sufficiently large, implies that $\bar{x}(t) \approx \bar{r}(t)$ for t sufficiently large (see (2.3.5)-(2.3.7)).

In view of (2.3.6) and (2.3.7), $\bar{x}(t) \approx \bar{r}(t)$ for t sufficiently large implies $\bar{u}(t) \approx v(t)$ for t sufficiently large. ■

Remark 17 When the opposite sign of the 'estimated' disturbance, which is $-\bar{u}(t) \approx -v(t)$, is fed back to system (2.2.5), the matched disturbance in system (2.2.5) will be cancelled out.

2.4.3 Simulation example

In this subsection, the adaptive algorithm Algorithm 1 and the results of Theorem 3 are demonstrated by numerical simulation. It will be recognized that the method of estimation and cancellation of disturbance is simple, but effective.

Configuration

The system, to be examined, is a second order single-input linear system expressed as follows:

$$\dot{r}(t) = Ar(t) + B(u(t) + d(t)),$$

where $d(t)$ represents external disturbance/uncertainty, and the system matrices are given by

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

An initial condition for the system is taken to be $r(t_0) = [3\ 0]^t$. For this problem, an accuracy specified by $\epsilon_e^2 = 2.0 \times 10^{-6}$ is required. For the adaptive algorithm, $\delta = -2$, $\kappa_2 = 5$, and $\omega = 10$ are used and initially, $\dot{\lambda}_1(t_0) = 0$ and $\lambda_1(t_0) = -2.0$ are set. For simulation purposes, the disturbance term is chosen to be $d(t) = 0.5 \sin t + 4r_1(t) + 2r_2(t)$, where $r_i(\cdot)$ are the components of $r(t) = [r_1(t)\ r_2(t)]^t$. Also, an estimated disturbance is used to cancel out effect of disturbance to the system; i.e. the opposite sign of the estimated disturbance is fed back to the system. The simulation has been performed with the following configuration:

Programing language: C++;

Compiler: g++ version 3.2.2;

Algorithm to obtain numerical solution of ODE: Runge-Kutta (see [31]);

Time step for Runge-Kutta algorithm: 0.0001.

Simulation results

The open-loop response of the states of the system are shown at Figures 2.4.1 and 2.4.2. From these figures, it is clear that, in the presence of disturbance, the open-loop response of the system is oscillating about the equilibrium state.

The closed-loop response of the states of the system are shown at Figure 2.4.3 and 2.4.4. From these figures, it is clear that, unlike the open-loop response, the states of the system converge to their equilibrium states. Thus, using the adaptive Algorithm 1 to estimate and cancel the disturbance, the effect of disturbance is eliminated from the closed-loop response of the system.

The actual and estimated disturbance are shown at Figure 2.4.5. The solid line represents the actual disturbance and the dashed line represents the estimated disturbance. In this figure, it is observed that the estimated disturbance converges to the actual one very rapidly. The error between the actual and estimated disturbances are shown in Figure 2.4.6. In this figure, it is seen that the difference between the actual and estimated one is very small. Thus, one can conclude that the accuracy and performance of estimation of disturbance are good for t sufficiently large.

For initial time interval, the actual and estimated disturbance are shown in Figure 2.4.7. The solid line represents the actual disturbance and the dashed line represents the estimated disturbance. In this figure, it is observed that for this initial time interval, the amplitude of the estimated disturbance is large compared with the amplitude of the actual disturbance. In practical situations, there will be constraints for input to systems. Hence, if an amplitude of an estimated disturbance is larger than an input constraint, it is not possible to feed back opposite sign of an estimated disturbance to a system to cancel out for this initial times. To overcome this problem, the adaptive algorithm will be required to be modified so that an estimation error of disturbance at this initial time becomes small enough. This problem is left for a future work.

The histories of the gains of the observer-like system and the eigenvalues of the error system are shown at Figure 2.4.8 and 2.4.9. In these figures, the eigenvalues decrease until they reach a certain value and then they remain constant. The histories of the value of the Lyapunov-like function are shown in Figure 2.4.10 and 2.4.11. Figure 2.4.10 represents overall history of the value of this function and Figure 2.4.11 represents behaviour of the value of this function at a later time. From Figure 2.4.10, it is clear that the value of this function decrease very rapidly. In Figure 2.4.11, it is seen that the value of this function remains less that the prescribed value; i.e. $\epsilon_e^2 = 2.0 \times 10^{-6}$. Therefore, one can conclude that for this configuration of the system and disturbance, the gains of the observer-like system, the eigenvalues of the error system, and the Lyapunov-like function have the properties described in Algorithm 1 and the lemmas introduced in the previous section.

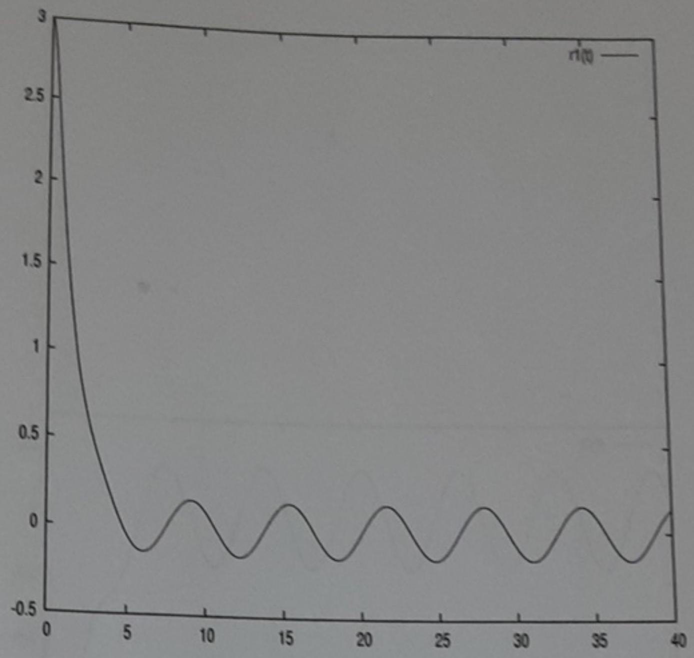

Figure 2.4.1: Open-loop response of the state $r_1(t)$.

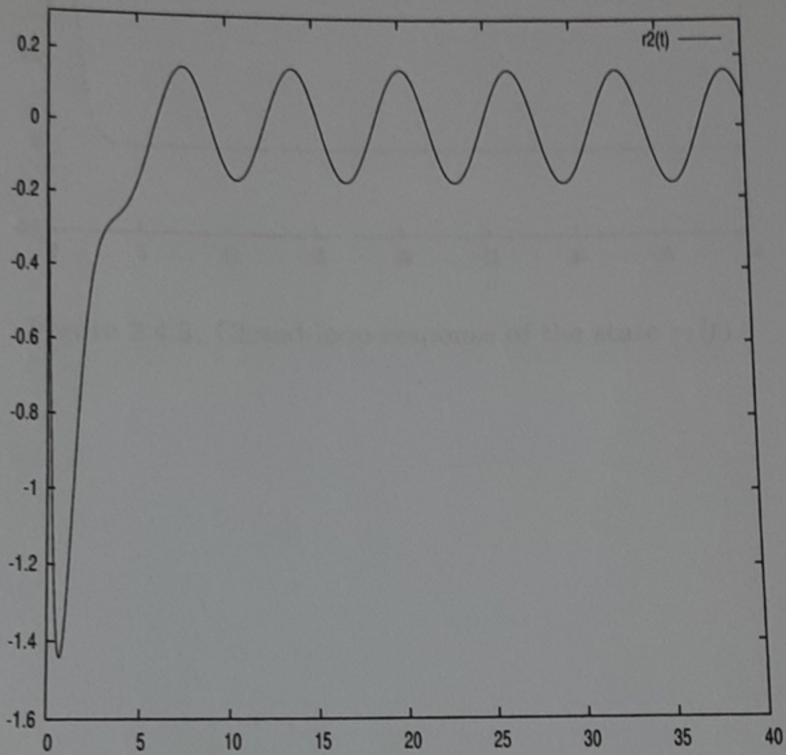

Figure 2.4.2: Open-loop response of the state $r_2(t)$.

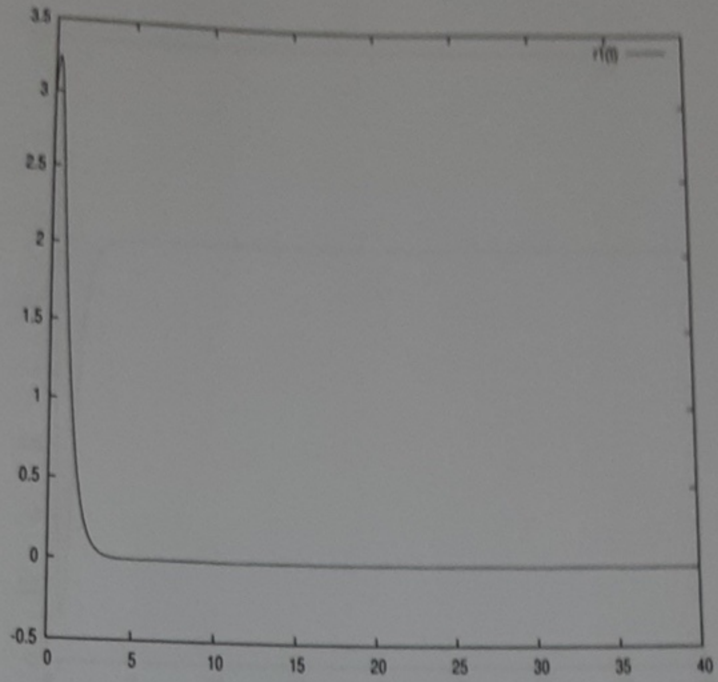

Figure 2.4.3: Closed-loop response of the state $r_1(t)$.

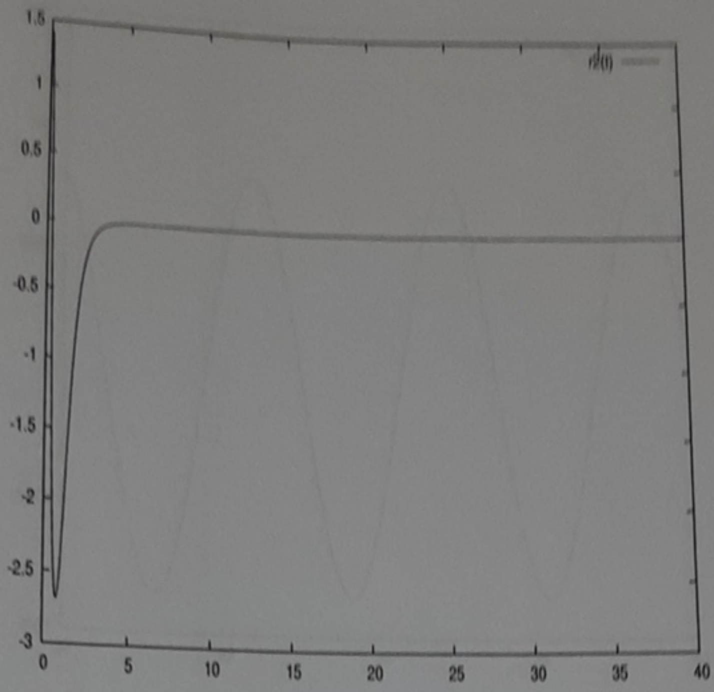

Figure 2.4.4: Closed-loop response of the state $r_2(t)$.

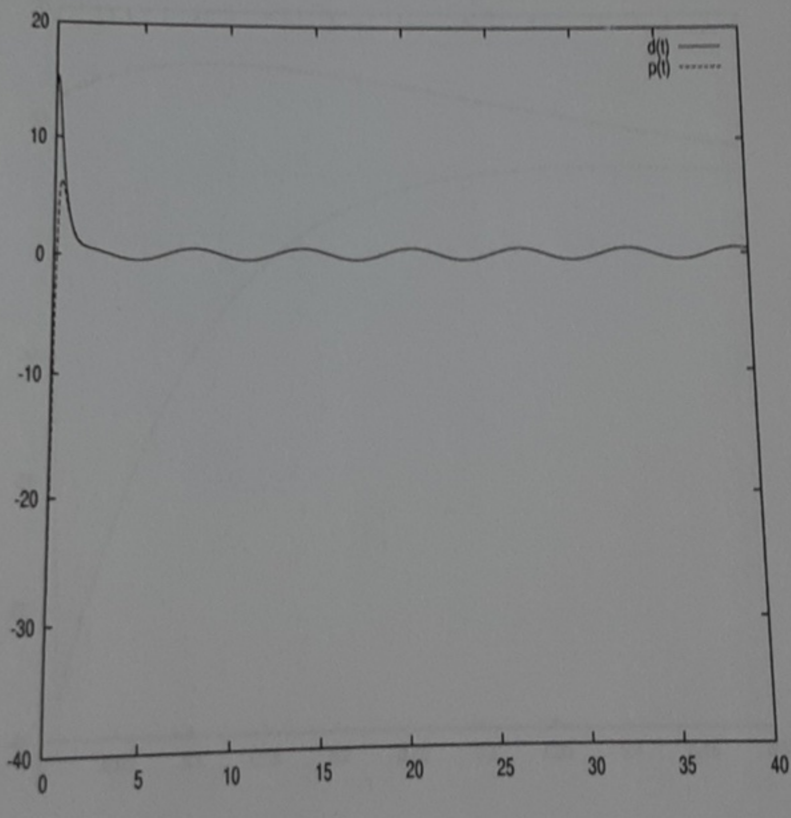

Figure 2.4.5: The actual and estimated disturbances, $d(t)$ and $p(t)$, respectively.

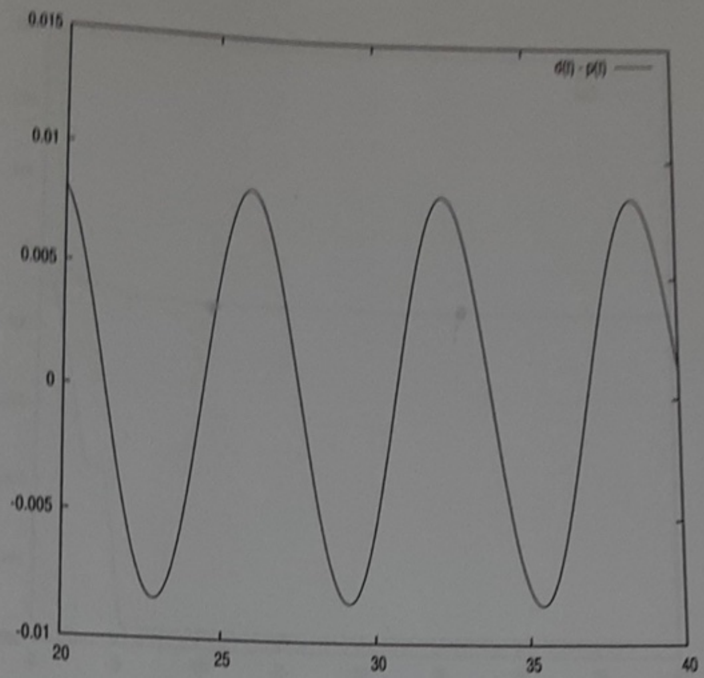

Figure 2.4.6: The difference between the actual and estimated disturbances at some later time interval.

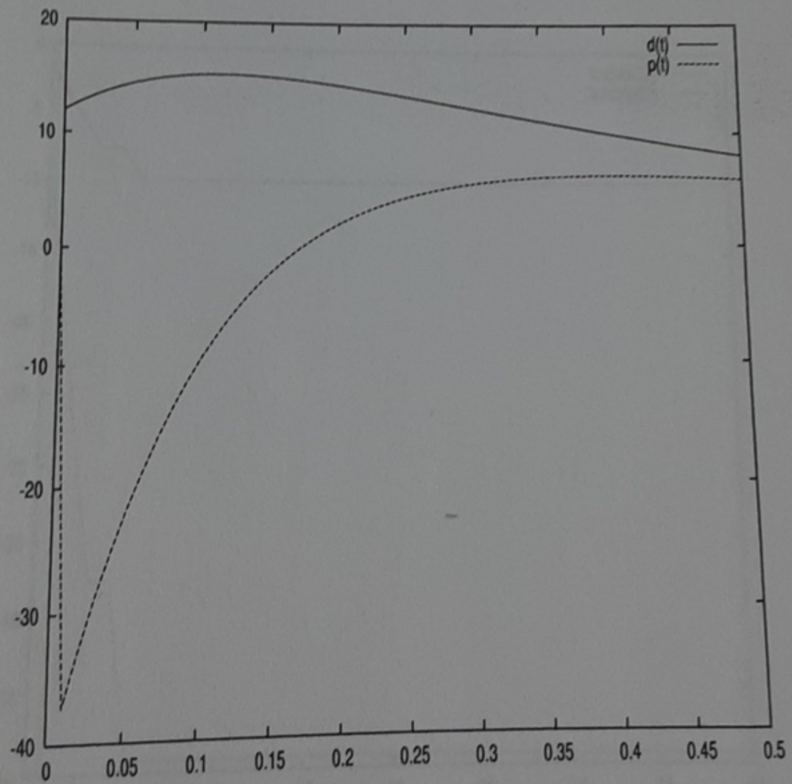

Figure 2.4.7: The actual and estimated disturbances, $d(t)$ and $p(t)$, respectively, at some initial time interval.

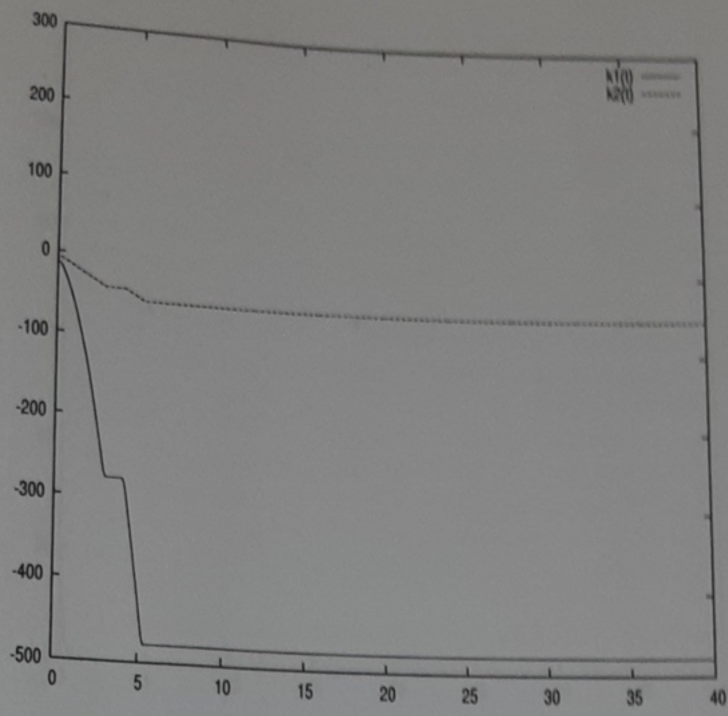

Figure 2.4.8: Histories of the feedback gains of the observer-like system: $k_1(t)$ and $k_2(t)$.

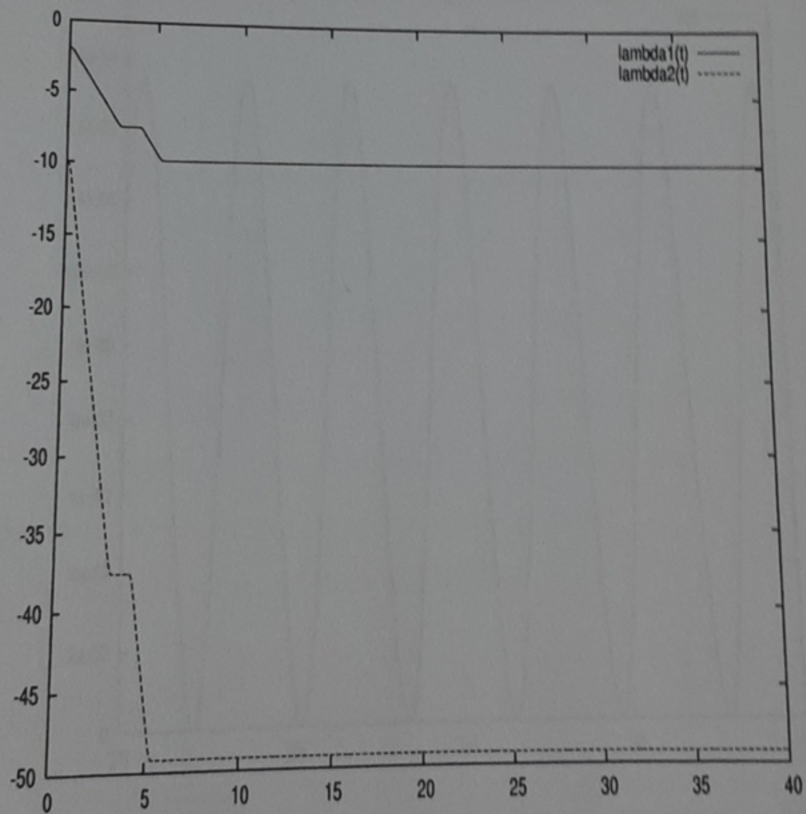

Figure 2.4.9: Histories of the eigenvalues of the error system: $\lambda_1(t)$ and $\lambda_2(t)$.

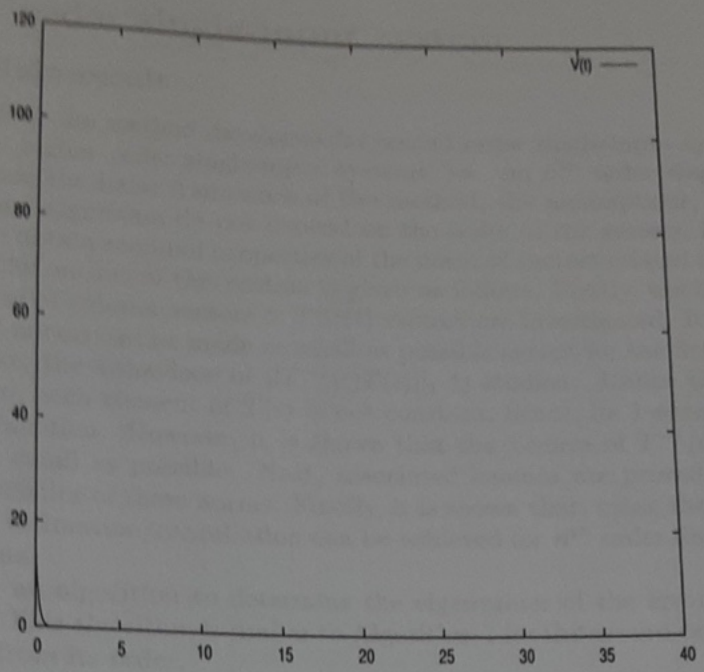

Figure 2.4.10: History of the Lyapunov-like function $V(t)$.

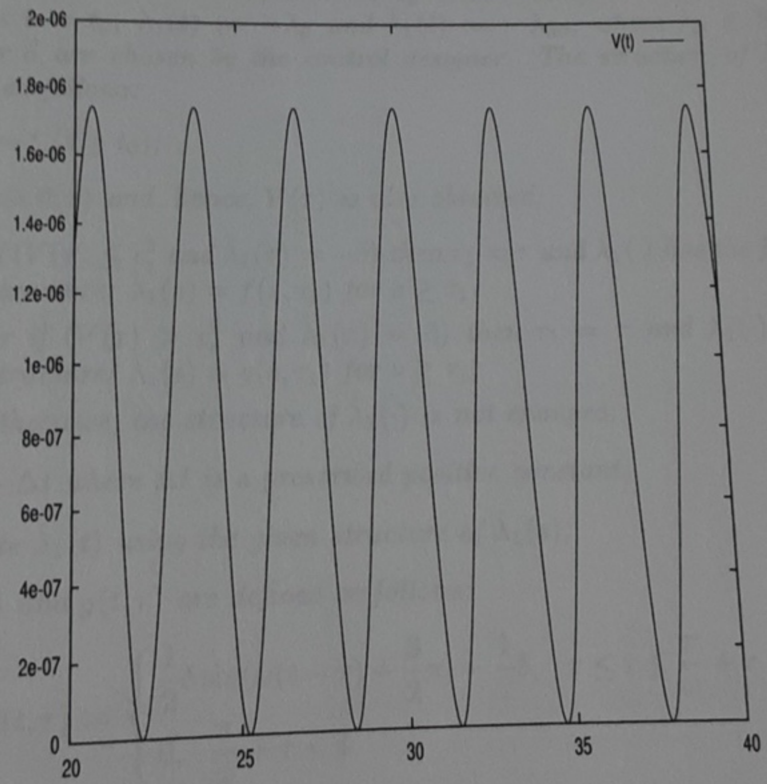

Figure 2.4.11: History of the Lyapunov-like function $V(t)$ at some later time interval.

2.5 n^{th} order single-input system

2.5.1 Main result

In this section, the method developed for second order single-input systems is extended to higher order single-input systems; i.e. an n^{th} order single-input system. Since the basic framework of the method, the assumptions, lemmas, theorems, and algorithm do not depend on the order of the system, it is only necessary to obtain required properties of the norm of vectors related to $T^{-1}(t)$ and $\dot{T}(t)$. The outline of this section is given as follows. Firstly, the behaviour of the 2-norm of column vectors of $T^{-1}(t)$ vectors are investigated. It is shown that these 2-norms can be made as small as possible except for the first column vector. Next, the behaviour of $||T^{-1}(t)\dot{T}(t)||_1$ is studied. Unlike the second order system, each element of $\dot{T}(t)$ is not constant, hence, its 1-norm is not a decreasing function. However, it is shown that the 1-norm of $T^{-1}(t)\dot{T}(t)$ can be made as small as possible. Next, associated lemmas are proved based on the characteristics of these norms. Finally, it is shown that, using these results, disturbance estimation/cancellation can be achieved for n^{th} order linear single-input systems.

Initially, an algorithm to determine the eigenvalues of the error system is introduced. This algorithm is similar to Algorithm 1 for the second-order system case, apart from its order.

Algorithm 2 *One of the eigenvalues of n^{th} order single-input error system (2.3.5), say $\lambda_1(t)$, is determined as follows. Suppose δ and ϵ_e are specified positive constants, which are determined by control designer. Define $V(t) := ||\bar{e}(t)||^2$. At $t = t_0$, $\lambda_1(t) := -\lambda_0$ and $\dot{\lambda}_1(t) := -\lambda_{d0}$, where $\lambda_0 \in \mathbb{R}^+$ and $\lambda_{d0} := 0$ or δ are chosen by the control designer. The structure of $\dot{\lambda}_1(t)$ is determined as follows:*

1. *let $\tau := t$ $(t \geq t_0)$;*
2. *evaluate $\bar{e}(\tau)$ and, hence, $V(\tau)$ is also obtained;*
3. (a) *if $(V(\tau) \leq \epsilon_e^2$ and $\dot{\lambda}_1(\tau) = -\delta)$ then $\tau_1 = \tau$ and $\dot{\lambda}_1(\cdot)$ has the following structure: $\dot{\lambda}_1(s) = f(s, \tau_1)$ for $s \geq \tau_1$;*
   (b) *or if $(V(\tau) > \epsilon_e^2$ and $\dot{\lambda}_1(\tau) = 0)$ then $\tau_1 = \tau$ and $\dot{\lambda}_1(\cdot)$ has the structure: $\dot{\lambda}_1(s) = g(s, \tau_1)$ for $s \geq \tau_1$;*
   (c) *otherwise, the structure of $\dot{\lambda}_1(\cdot)$ is not changed;*
4. *$t = t + \Delta t$ where Δt is a prescribed positive constant;*
5. *evaluate $\dot{\lambda}_1(t)$ using the given structure of $\dot{\lambda}_1(s)$;*

where $f(t, \tau)$ and $g(t, \tau)$ are defined as follows:

$$f(t,\tau) := \begin{cases} \frac{1}{2}\delta \sin(\omega(t-\tau) + \frac{3}{2}\pi) - \frac{1}{2}\delta, & \tau \leq t \leq \frac{\pi}{\omega} + \tau \\ 0, & \frac{\pi}{\omega} + \tau < t \end{cases}$$

$$g(t,\tau) := \begin{cases} \frac{1}{2}\delta \sin(\omega(t-\tau) + \frac{1}{2}\pi) - \frac{1}{2}\delta, & \tau \leq t \leq \frac{\pi}{\omega} + \tau \\ -\delta, & \frac{\pi}{\omega} + \tau < t \end{cases}$$

where ω is specified constant.

The remaining eigenvalues of the error system corresponding to an n^{th} order single-input system, say $\lambda_2(t)\cdots\lambda_n(t)$, are determined as follows:

$$\lambda_i(t) = \kappa_i\lambda_1(t),$$

where κ_i $(i = 2\cdots n)$ are prescribed positive constants determined by control designer, κ_i satisfy $\kappa_i \neq 1$ for all i, with $\kappa_i \neq \kappa_j$ for $i \neq j$.

Remark 18 As stated as before, the system and input matrices are assumed to be transformed into controllable canonical form as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{2.5.1}$$

Remark 19 As in the case of the second-order system, since, the n^{th} order system is assumed to be represented by controllable canonical form, the feedback gains for the observer-like system are determined in a straightforward manner. The feedback gains of the n^{th} order observer-like system (2.2.11) are determined as follows:

$$\begin{aligned} k_n(t) &= -a_n - \sum_{i=1}^{n}(-\lambda_i(t)) \qquad (2.5.2) \\ k_{n-1}(t) &= -a_{n-1} - \sum_{i=j=1, i\neq j}^{i=j=n}(-\lambda_i(t))(-\lambda_j(t)) \\ k_{n-2}(t) &= -a_{n-2} - \sum_{i=j=k=1, i\neq j\neq k}^{i=j=k=n}(-\lambda_i(t))(-\lambda_j(t))(-\lambda_k(t)) \\ &\vdots \\ k_1(t) &= -a_1 - \Pi_{i=1}^{n}(-\lambda_i(t)). \qquad (2.5.3) \end{aligned}$$

Remark 20 T matrix, which is known as the Vandermonde matrix, for the n^{th} order system is given as follows:

$$T(t) = \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1(t) & \cdots & \lambda_n(t) \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1}(t) & \cdots & \lambda_n^{n-1}(t) \end{bmatrix}. \tag{2.5.4}$$

In the following lemmas, characteristics of norms relating to $T^{-1}(t)$ are investigated. There are some results relating to the inverse of the Vandermonde matrix. In [30], inverse of T has been obtained. An upper bound, lower bound, and equality of ∞ − norm of inverse of the Vandermonde matrix, which is defined as $||A||_\infty = \max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}|$, have been investigated in [33] and [34]. In this study, a different approach is taken to investigate characteristics of norms relating to $T^{-1}(t)$, those are $||T^{-1}(t)B||$ and $||T^{-1}(t)\dot{T}(t)||_1$. Since the basic framework of analysis of the following lemmas does not require that $T(t)$ should be the Vandermonde matrix, it is possible to investigate characteristics of norms relating to $T(t)$ for a multi-input system, which is not a Vandermonde matrix anymore, using a similar analysis to that for the following lemmas.

Lemma 11 *In $\mathbb{R}^n$, if a vector $a(t)$ is orthogonal to each of given vectors $b_i(t)$ $(i = 1, \cdots, n-1)$ for all t, and if $b_i(t)$ and $b_j(t)$ $(i \neq j)$ are linearly independent, and if the 'direction' of each $b_i(t)$ is fixed for all t, then 'direction' of $a(t)$ is uniquely determined for each time instant and the 'direction' of $a(t)$ is fixed for all t.*

Proof The statement of this lemma is proved by contradiction argument. Assume there exists a vector $\bar{a}(t)$ such that it is orthogonal to each $b_i(t)$ $(i = 1, \cdots, n-1)$ and the 'direction' of $\bar{a}(t)$ is different from the 'direction' of $a(t)$; i.e. there does not exist a scalar function $x(t)$ such that the relation $\bar{a}(t) = x(t)a(t)$ holds. Since, the 'direction' of each $b_i(t)$ is fixed for all t, there exists scalar functions $y_i(t)$ and constant vectors $\bar{b}_i$ such that the equations $b_i(t) = y_i(t)\bar{b}_i$ hold for $i = 1, \cdots, n-1$. Since, $a(t)$ is orthogonal to each $b_i(t)$, $a(t)$ is also orthogonal to each $\bar{b}_i$. Since, $a(t)$ is orthogonal to each $\bar{b}_i$ in $\mathbb{R}^n$, vectors $\{\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{n-1}, a\}$ is a basis for $\mathbb{R}^n$. Hence, there exist scalars α_i $(i = 1, \cdots, n)$ such that following relation holds:

$$\bar{a}(t) = \alpha_1\bar{b}_1 + \alpha_2\bar{b}_2 + \cdots + \alpha_{n-1}\bar{b}_{n-1} + \alpha_n a.$$

Applying $\bar{a}(t)^t$ from left hand side gives

$$\bar{a}(t)^t\bar{a}(t) = \alpha_1\bar{a}(t)^t\bar{b}_1 + \alpha_2\bar{a}(t)^t\bar{b}_2 + \cdots + \alpha_{n-1}\bar{a}(t)^t\bar{b}_{n-1} + \alpha_n\bar{a}(t)^t a(t). \quad (2.5.5)$$

Since, $\bar{a}(t)$ is orthogonal to each $\bar{b}_i$, $\bar{a}(t)^t\bar{b}_i = 0$ for $i = 1, \cdots, n-1$. Hence, (2.5.5) is expressed as follows:

$$\bar{a}(t)^t\bar{a}(t) = \alpha_n\bar{a}(t)^t a(t).$$

It follows that

$$\bar{a}(t) = \alpha_n a(t). \quad (2.5.6)$$

(2.5.6) implies that the 'direction' of $a(t)$ is the same as that of $\bar{a}(t)$. This contradicts original assumption. Thus, at each time instant, the direction of $a(t)$ is uniequly determined.

Next, assume that the direction of $a(t)$ is time-varying. Thus, there exists t_1 and t_2 such that the 'direction' of $a(t_1)$ is different from that of $a(t_2)$ and both $a(t_1)$ and $a(t_2)$ are orthogonal to each $b_i(t)$. This leads contradiction of the assumption by the same reason as above. Then, the result follows. ■

Lemma 12 *If the 'directions' of $a(t)$ and $b(t)$ are fixed for all t, the magnitude of $\frac{a(t)\cdot b(t)}{||a(t)|| \, ||b(t)||}$ is constant for all t.*

Proof Since the 'directions' of $a(t)$ and $b(t)$ are not changed for all t, there exist scalar functions $x(t)$ and $y(t)$ and constant vector $\bar{a}$ and $\bar{b}$ such that following relation holds:

$$a(t) := x(t)\bar{a},$$
$$b(t) := y(t)\bar{b}.$$

Hence,

$$\frac{a(t) \cdot b(t)}{||a(t)|| \, ||b(t)||} = \frac{|x(t)y(t)\,\bar{a} \cdot \bar{b}|}{|x(t)|\,|y(t)|\,||\bar{a}||\,||\bar{b}||}$$
$$= \frac{|\bar{a} \cdot \bar{b}|}{||\bar{a}||\,||\bar{b}||}. \quad (2.5.7)$$

It is clear that right hand side of (2.5.7) is constant. Then, result follows. ■

Lemma 13 *The 2-norm of the n^{th} column vector of T^{-1} matrix is a non-increasing function, when $\lambda_i(t)$ are determined by Algorithm 2.*

Proof The matrix $T(t)$, given in (2.5.4), can be expressed in the form

$$T(t) = \begin{bmatrix} t_1(t) \\ \vdots \\ t_n(t) \end{bmatrix},$$

where $t_i(t)$ is i^{th} row vector of $T(t)$. In addition, $T^{-1}(t)$ can be represented by:

$$T^{-1}(t) = \begin{bmatrix} \bar{t}_1(t) & \cdots & \bar{t}_n(t) \end{bmatrix},$$

where $\bar{t}_i(t)$ is the i^{th} column vector of $T^{-1}(t)$. The i^{th} row vector of $T(t)$ has the following structure:

$$t_i(t) = \begin{bmatrix} \lambda_1^{i-1}(t) & \lambda_2^{i-1}(t) & \cdots & \lambda_n^{i-1}(t) \end{bmatrix}.$$

By Algorithm 2, the relation $\lambda_i(t) = \kappa_i \lambda_1(t)$ holds. Hence,

$$\begin{aligned} t_i(t) &= \begin{bmatrix} \lambda_1^{i-1}(t) & \kappa_2^{i-1}\lambda_1^{i-1}(t) & \cdots & \kappa_n^{i-1}\lambda_1^{i-1}(t) \end{bmatrix} \\ &= \lambda_1^{i-1}(t) \begin{bmatrix} 1 & \kappa_2^{i-1} & \cdots & \kappa_n^{i-1} \end{bmatrix}. \end{aligned}$$

Thus, the 'direction' of each $t_i(t)$ is fixed for all t. By definition,

$$T(t)T^{-1}(t) = I,$$

where I is the $n \times n$ identity matrix, and, hence,

$$\begin{aligned} t_1(t)\bar{t}_n(t) &= 0 \\ &\vdots \\ t_{n-1}(t)\bar{t}_n(t) &= 0 \\ t_n(t)\bar{t}_n(t) &= 1. \end{aligned} \quad (2.5.8)$$

These relations mean that $\bar{t}_n(t)$ is orthogonal to the vectors $\{t_1(t) \cdots t_{n-1}(t)\}$, but not orthogonal to $t_n(t)$. It is already shown that the 'direction' of each $t_i(t)$ is fixed for all t. Thus, by Lemma 11, the 'direction' of $\bar{t}_n(t)$ is uniquely determined and the 'direction' of $\bar{t}_n(t)$ is fixed for all t. Since the 'directions' of $t_n(t)$ and $\bar{t}_n(t)$ are fixed for all t, by Lemma 12, the magnitude of $\frac{t_n(t)\,\bar{t}_n(t)}{||t_i(t)||\,||\bar{t}_n(t)||}$

is constant for all t. By (2.5.8), $t_n(t)\bar{t}_n(t) = ||t_n(t)||||\bar{t}_n(t)||\frac{t_n(t)\bar{t}_n(t)}{||t_i(t)||||\bar{t}_n(t)||} = 1$. Therefore, there exists a constant α_n such that

$$
\begin{aligned}
||\bar{t}_n(t)|| &= \frac{1}{\alpha_n}\frac{1}{||t_n(t)||} \\
&= \frac{1}{\alpha_n}\frac{1}{\sqrt{\lambda_1^{2(n-1)}(t)+\cdots+\lambda_n^{2(n-1)}(t)}}.
\end{aligned} \tag{2.5.9}
$$

Since $|\lambda_i(t)|$ is a non-decreasing function, $||\bar{t}_n(t)||$ is a non-increasing function. ■

Lemma 14 *The 2-norm of i^{th} column vector of $T^{-1}(t)$, $i > 1$, is a non-increasing function where $\lambda_i(t)$ are determined by Algorithm 2.*

Proof The analysis in the proof of Lemma 13 does not depend on the index n, namely n can be replaced by 1, $\cdots$, or n − 1. Hence,

$$
||\bar{t}_i(t)|| = \frac{1}{\alpha_i}\frac{1}{\sqrt{\lambda_1^{2(i-1)}(t)+\cdots+\lambda_n^{2(i-1)}(t)}},
$$

where α_i is some constant. Hence, for $i > 1$, $t \mapsto ||\bar{t}_i(t)||$ is non-increasing function. ■

Lemma 15 *If $\lambda_i(t)$ are determined by Algorithm 2, then $t \mapsto ||T^{-1}(t)B||$ is a non-increasing function.*

Proof By definition of B, see (2.5.1), it follows that

$$
T^{-1}B = \bar{t}_n(t), \tag{2.5.10}
$$

where $\bar{t}_n(t)$ is the n^{th} column vector of $T^{-1}(t)$, and, hence, $||T^{-1}B(t)|| = ||\bar{t}_n(t)||$. By Lemma 14, the result follows. ■

Lemma 16 *If Algorithm 2 is used, $t \mapsto ||T^{-1}(t)\dot{T}(t)||_1$ is non-increasing function.*

Proof Recall that $T(t)$ and $T^{-1}(t)$ have the following structure:

$$
\begin{aligned}
T(t) &= \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1(t) & \cdots & \lambda_n(t) \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1}(t) & \cdots & \lambda_n^{n-1}(t) \end{bmatrix}, \\
T^{-1}(t) &= \begin{bmatrix} \bar{t}_1(t) & \cdots & \bar{t}_n(t) \end{bmatrix} \\
||\bar{t}_i(t)|| &= \frac{1}{\alpha_i}\frac{1}{\sqrt{\lambda_1^{2(i-1)}(t)+\cdots+\lambda_n^{2(i-1)}(t)}},
\end{aligned} \tag{2.5.11}
$$

where $\bar{t}_i(t)$ is the i^{th} column vector of $T^{-1}(t)$ and α_i $(i = 1\cdots n)$ are constant. Each element of the $\bar{t}_i$ vector should be less than or equal to the right hand side of (2.5.11). Hence, each element of $T^{-1}(t)$, namely $\tilde{t}_{ij}$, is represented by

$$
|\tilde{t}_{ij}(t)| = \xi_{ij}(t)\frac{1}{\sqrt{\lambda_1^{2(j-1)}(t)+\cdots+\lambda_n^{2(j-1)}(t)}},
$$

where $\xi_{ij}(t)$ satisfies

$$0 < \xi_{ij}(t) \leq \frac{1}{\alpha_j}, \quad \forall t. \tag{2.5.12}$$

Consider the vector

$$a_i(t) = \left[\begin{array}{cccc} \xi_{i1}(t)\frac{1}{n} & \xi_{i2}(t)\frac{1}{\sqrt{\lambda_1^2(t)+\cdots\lambda_n^2(t)}} & \cdots & \xi_{in}(t)\frac{1}{\sqrt{\lambda_1^{2(n-1)}(t)+\cdots\lambda_n^{2(n-1)}(t)}} \end{array} \right]$$

where each element of the a_i vector is the magnitude of each element of the i^{th} row vector of $T^{-1}(t)$.

Now, $\dot{T}(t)$ is given by

$$\dot{T}(t) = \begin{bmatrix} 0 & \cdots & 0 \\ \dot{\lambda}_1(t) & \cdots & \dot{\lambda}_n(t) \\ \vdots & \ddots & \vdots \\ (n-1)\dot{\lambda}_1(t)\lambda_1^{n-2}(t) & \cdots & (n-1)\dot{\lambda}_n(t)\lambda_n^{n-2}(t) \end{bmatrix}.$$

Hence, the j^{th} column vector of $\dot{T}(t)$ is

$$b_j(t) = \begin{bmatrix} 0 \\ \dot{\lambda}_j(t) \\ \vdots \\ (n-1)\dot{\lambda}_j(t)\lambda_j^{n-2}(t) \end{bmatrix}.$$

The scalar product of $a_i(t)$ and $b_j(t)$ is

$$\begin{aligned} c_{ij}(t) = a_i(t)\cdot b_j(t) = 0 + \xi_{i2}(t)\dot{\lambda}_j(t)\frac{1}{\sqrt{\lambda_1^2(t)+\cdots\lambda_n^2(t)}} + \cdots \\ + (n-1)\xi_{in}(t)\dot{\lambda}_j(t)\frac{\lambda_j^{n-2}(t)}{\sqrt{\lambda_1^{2(n-1)}(t)+\cdots\lambda_n^{2(n-1)}(t)}}, \end{aligned} \tag{2.5.13}$$

where $c_{ij}(t)$ represents the magnitude of the i^{th} row j^{th} column element of $T^{-1}(t)\dot{T}(t)$. The general term in the right hand side of (2.5.13) has the form

$$(k-1)\xi_{ik}(t)\dot{\lambda}_j(t)\frac{\lambda_j^{k-2}(t)}{\sqrt{\lambda_1^{2(k-1)}(t)+\cdots+\lambda_n^{2(k-1)}(t)}}. \tag{2.5.14}$$

Since $\frac{|\lambda_j^{k-2}(t)|}{\sqrt{\lambda_1^{2(k-1)}(t)+\cdots+\lambda_n^{2(k-1)}(t)}}$ is a non-increasing function, $\dot{\lambda}_j(t)$ is bounded and the inequality (2.5.12) holds, then the magnitude of the every term in the right hand side of (2.5.13) is a non-increasing function. Then, the result follows. ■

Remark 21 By using Lemmas 15 and 16, all the results of the analysis of second order systems can be applied to n^{th} order systems. Recall that the time derivative of the function $V(t)$ along solutions to (2.3.5) satisfies

$$\dot{V}(t) \leq \beta(t)||\bar{e}(t)||^2 + \gamma(t)||\bar{e}(t)||,$$

where

$$\beta(t) = 2(\sigma_{\max}(\Lambda(t)) + \|\dot{T}(t)\|\|T^{-1}(t)\|)$$

and

$$\gamma(t) = 2\alpha\|T^{-1}(t)\|_1\|B\|.$$

For the second order system,

1. $\gamma(t) > 0$;

2. $\beta(t)$ is a non-increasing function;

3. $\gamma(t)$ is a decreasing function.

For the n^{th} order system, $\beta(t)$ and $\gamma(t)$ are replaced by

$$\beta(t) = 2(\sigma_{\max}(\Lambda(t)) + \|T^{-1}(t)\dot{T}(t)\|_1) \tag{2.5.15}$$

and

$$\gamma(t) = 2\alpha\|T^{-1}(t)B\|. \tag{2.5.16}$$

As a consequence of Lemmas 15 and 16, $\|T^{-1}(t)\dot{T}(t)\|_1$ and $\|T^{-1}(t)B\|$ are non-increasing functions. Thus, the results that apply to a second order system can be applied to a n^{th} order system.

Remark 22 In view of Remark 21, $\epsilon(t)$ for the n^{th}order system are given by

$$\epsilon(t) = -\frac{\alpha\|T^{-1}(t)B\|}{\sigma_{\max}(\Lambda(t)) + \|T^{-1}(t)\dot{T}(t)\|_1}. \tag{2.5.17}$$

Theorem 4 *For the n^{th} order single-input system (2.2.5), $\bar{u}(t) \approx p(t)$ can be achieved for t sufficiently large, where p(t) is the matched disturbance/uncertainty in system (2.2.5) and $\bar{u}(t)$ is the control input to the observer-like system (2.2.4), using observer-like system (2.2.4), feedforward filter (2.2.8), modified reference signal (2.2.9) and the feedback control defined by (2.5.2)-(2.5.3) and Algorithm2.*

Proof In the view of remark 21, the proof is an immediate consequence of Lemmas 15 and 16. ■

Remark 23 As well as second order single-input system, if the opposite sign of the 'estimated' disturbance, which is $-\bar{u}(t) \approx -p(t)$, is fed back to system (2.2.5), the matched disturbance in system (2.2.5) will be cancelled out.

2.5.2 Simulation example

In this subsection, theorems and algorithms developed in Section 2.5.1 are demonstrated by numerical simulations.

Configuration

The system to be examined is a third order single-input system given by

$$\dot{r}(t) = Ar(t) + B(d(t) + u(t)),$$

where $d(t)$ denotes the matched uncertainty/disturbance, and $u(t)$ represents the control input to the system. The system matrix and input matrix are given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -30 & -31 & -10 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and the dynamics are subject to the initial condition, $x(t_0) = [3\ 0\ 0]^t$. Also, $\delta = 2.0$, $\kappa_2 = 1.5$, $\kappa_3 = 2.5$, $\epsilon_e^2 = 2.0 \times 10^{-6}$, $\omega = 10$, $\lambda_1(t_0) = -2.0$, and $\dot{\lambda}_i(t_0) = 0$ are chosen for adaptive algorithm. For the simulation purposes, the disturbance is chosen to be $d(t) = \sin t + 24r_1(t) + 20r_2(t) + 4r_3(t)$, where $r_i(\cdot)$ are components of $r(t) = [r_1(t)\ r_2(t)\ r_3(t)\ r_4(t)]^t$. Also, an estimated disturbance is used to cancel out effect of disturbance to the system; i.e. the opposite sign of the estimated disturbance is fed back to the system. The simulation program has been implemented using the following elements:

Programing language: C++;

Compiler: g++ version 3.2.2;

Algorithm to obtain numerical solution of ODE: Runge-Kutta (see [31]);

Time step for Runge-Kutta: 0.0001;

Algorithm to obtain inverse matrix: Gauss-Jordan Elimination (see [31]).

Simulation results

The open-loop responses of the states of the system are shown in Figures 2.5.1, 2.5.2, and 2.5.3. From these figures, it is clear that, in the presence of disturbance, the states are perturbed from their equilibriums.

The closed-loop response of the states of the system are shown at Figures 2.5.4, 2.5.5, and 2.5.6. From these figures, it is clear that, using the estimation and cancellation of the disturbance technique, the states converge to their equilibriums.

The actual and estimated disturbances are shown in Figure 2.5.7. The solid line represents the disturbance and the dashed line represents estimated disturbance. In this figure, it is observed that estimated disturbance converges to true disturbance very rapidly. The difference between the true and estimated disturbances, at some later time period, is shown in Figure 2.5.8. It is observed in Figure 2.5.8 that the error between the estimated disturbance and the true disturbance is very small. Thus, it is concluded that estimation and cancellation of the disturbance has been successfully achieved and its performance is good.

The eigenvalues of the error system and the feedback gains of the observer-like system are shown in Figure 2.5.9 and 2.5.10. In these figures, it is observed that the values are decreased until they reach a certain value and they remain at those values. The history of the Lyapunov-like function is shown in Figure 2.5.11. From this figure, it can been seen that the value of this function decreases very rapidly. The behaviour of the value of this function at a later time is shown in Figure 2.5.12. In this figure, the value of this function is bounded by the given constant; i.e. $\epsilon_e^2 = 2.0 \times 10^{-6}$. Therefore, it is concluded that the eigenvalues of the error system, the feedback gains of the observer-like system and the value of the Lyapunov-like function behave as described by the lemmas developed in the Section 2.5.1.

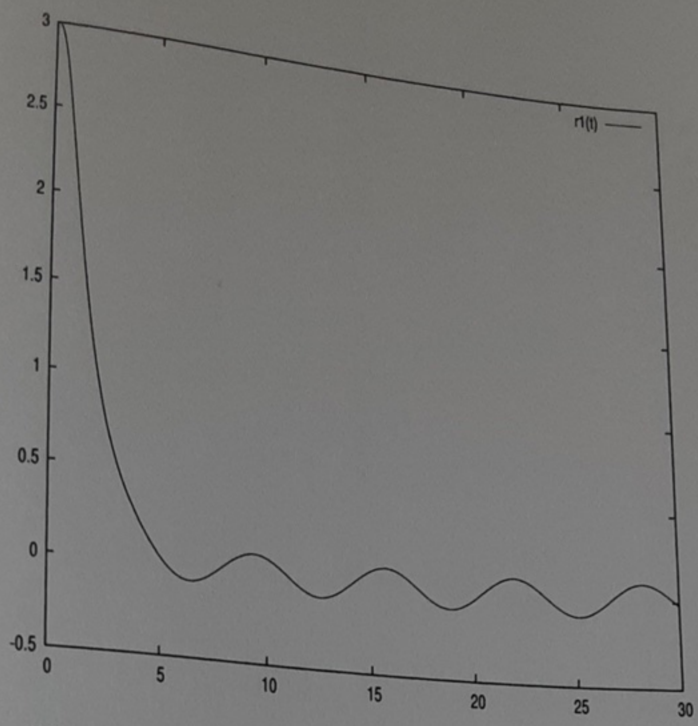

Figure 2.5.1: Open-loop response of the state $r_1(t)$.

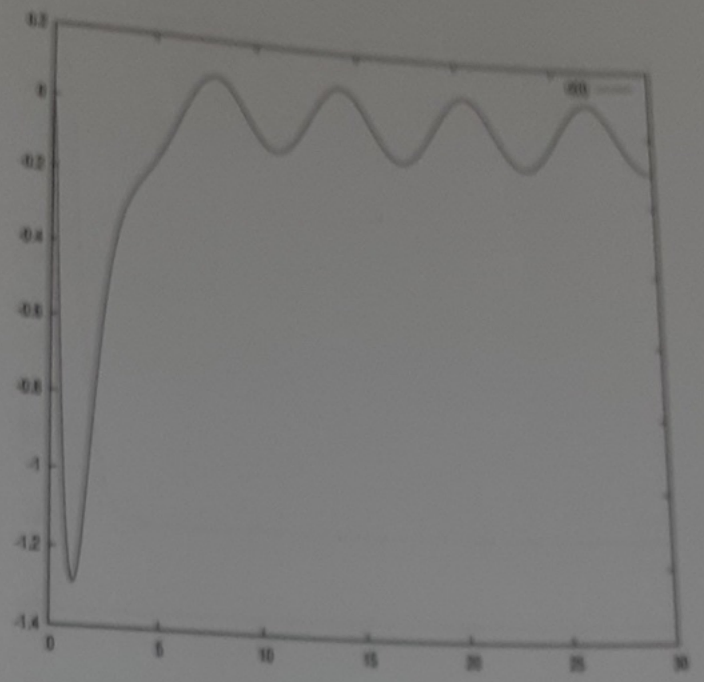

Figure 2.5.2: Open-loop response of the state $r_2(t)$.

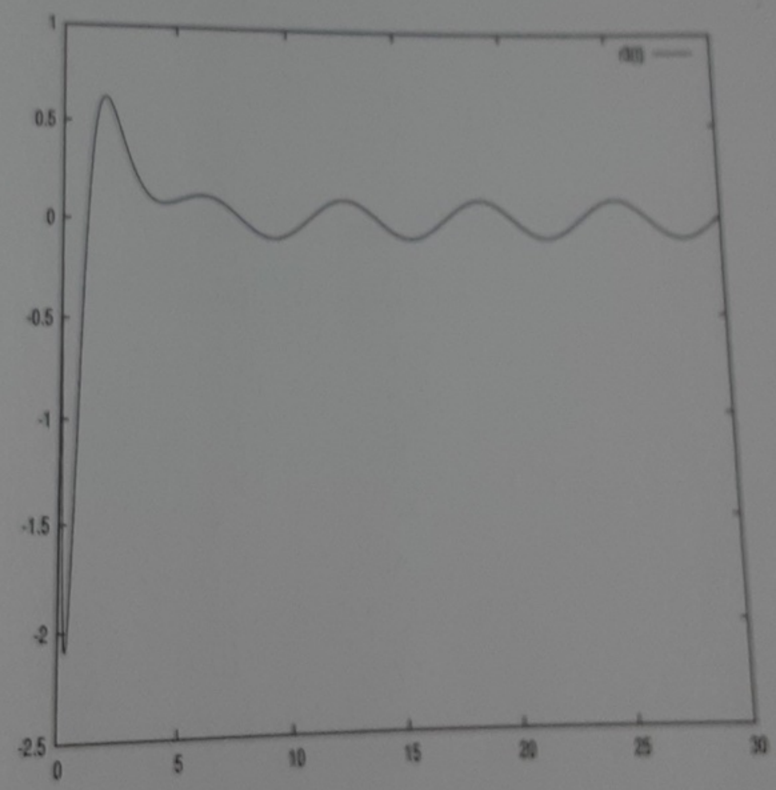

Figure 2.5.3: Open-loop response of the state $r_3(t)$.

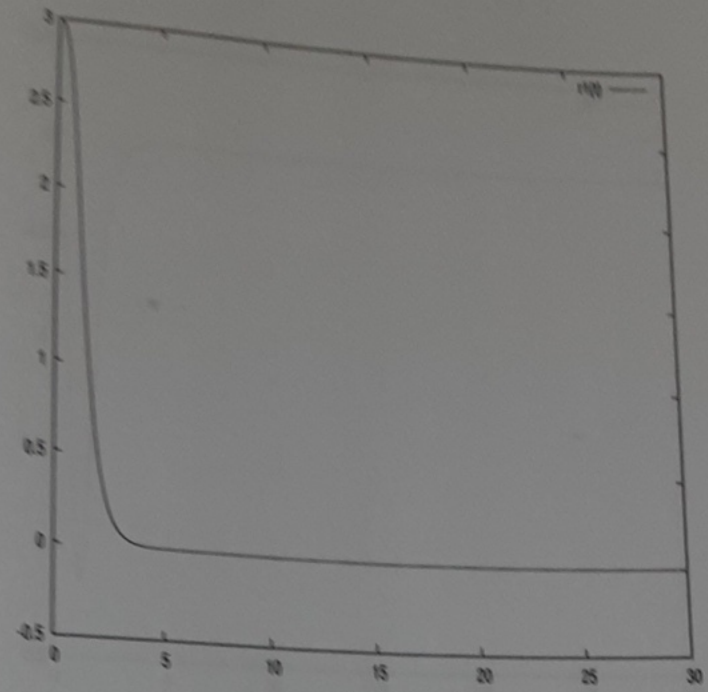

Figure 2.5.4: Closed-loop response of the state $r_1(t)$.

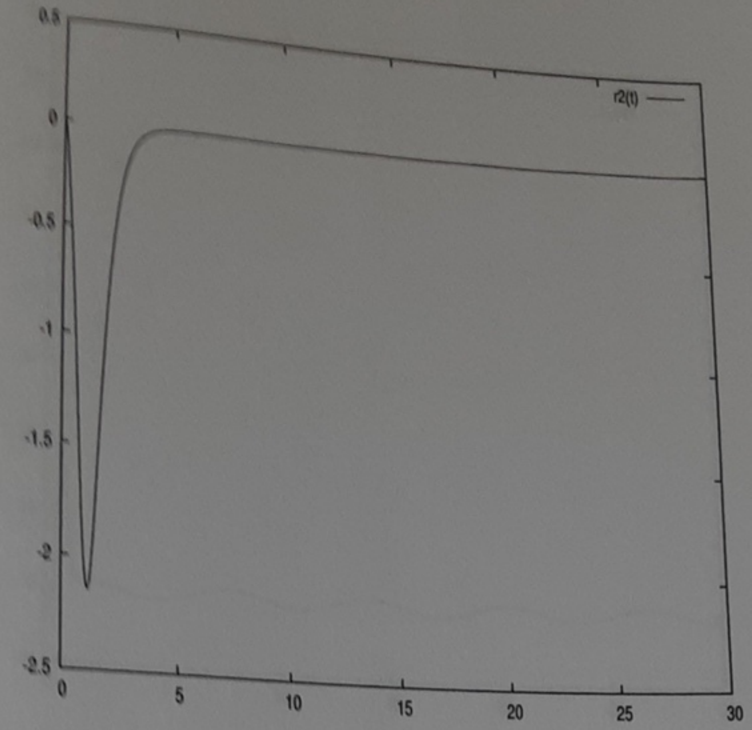

Figure 2.5.5: Closed-loop response of the state $r_2(t)$.

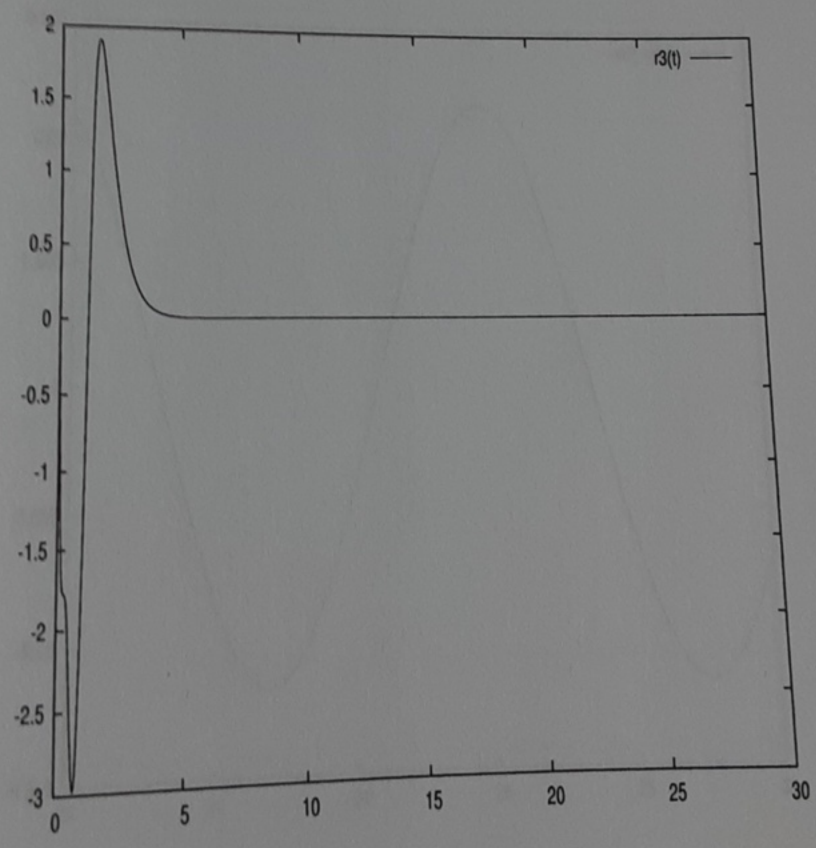

Figure 2.5.6: Closed-loop response of the state $r_3(t)$.

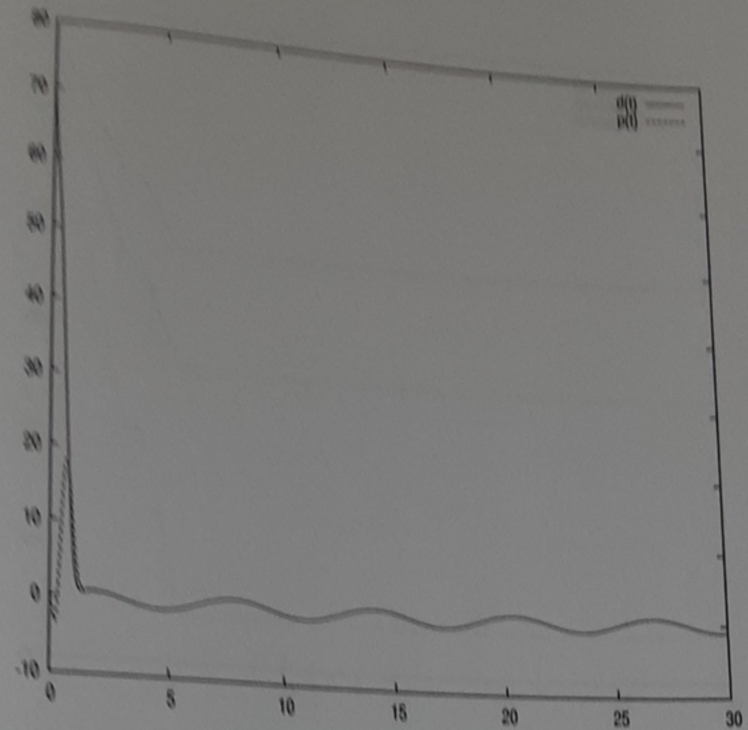

Figure 2.5.7: The actual and estimated disturbances, $d(t)$ and $p(t)$, respectively.

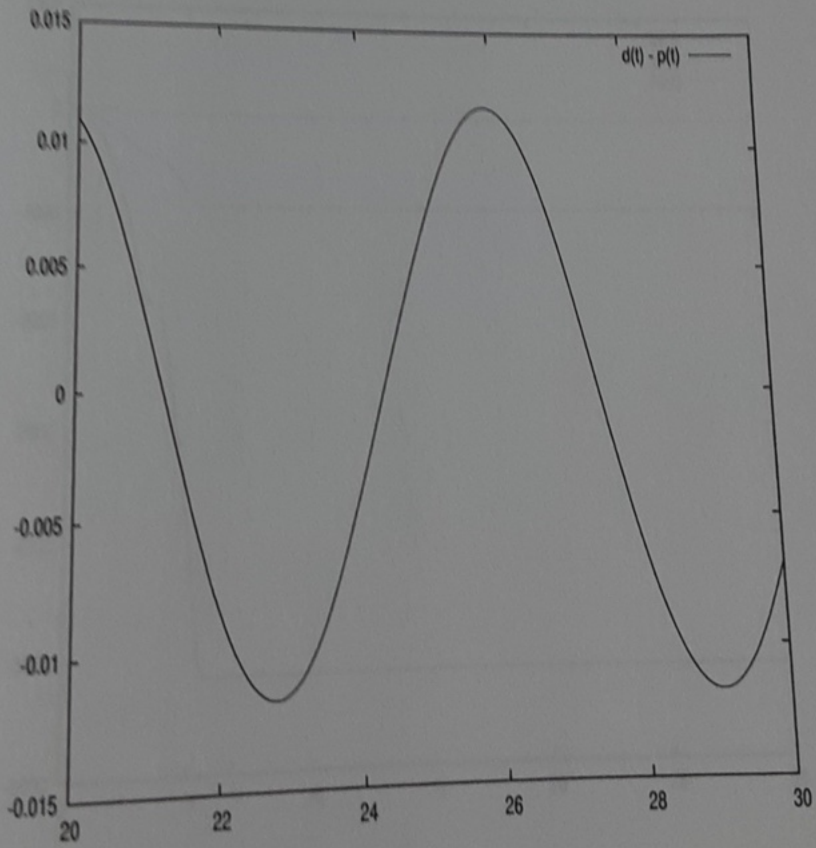

Figure 2.5.8: Estimation error of disturbance, $d(t) - p(t)$, at some later interval.

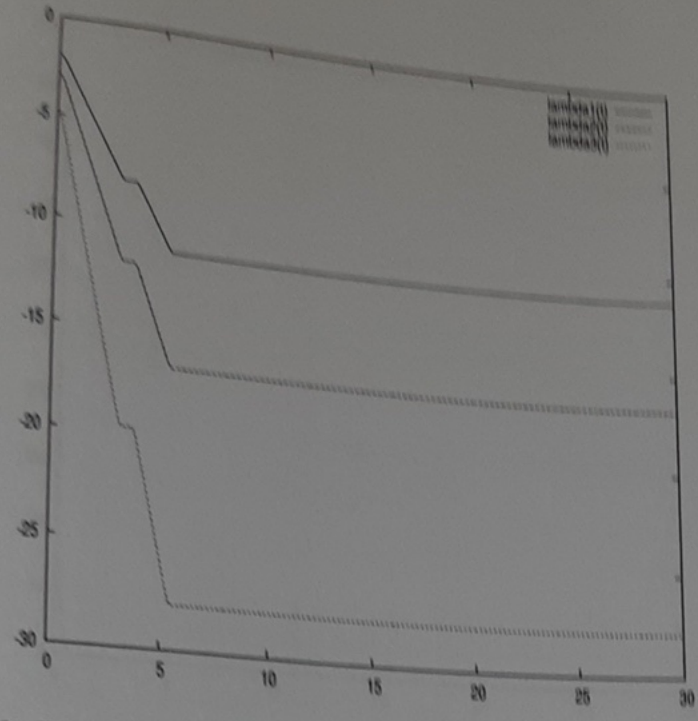

Figure 2.5.9: Histories of the eigenvalues of the error system: $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$.

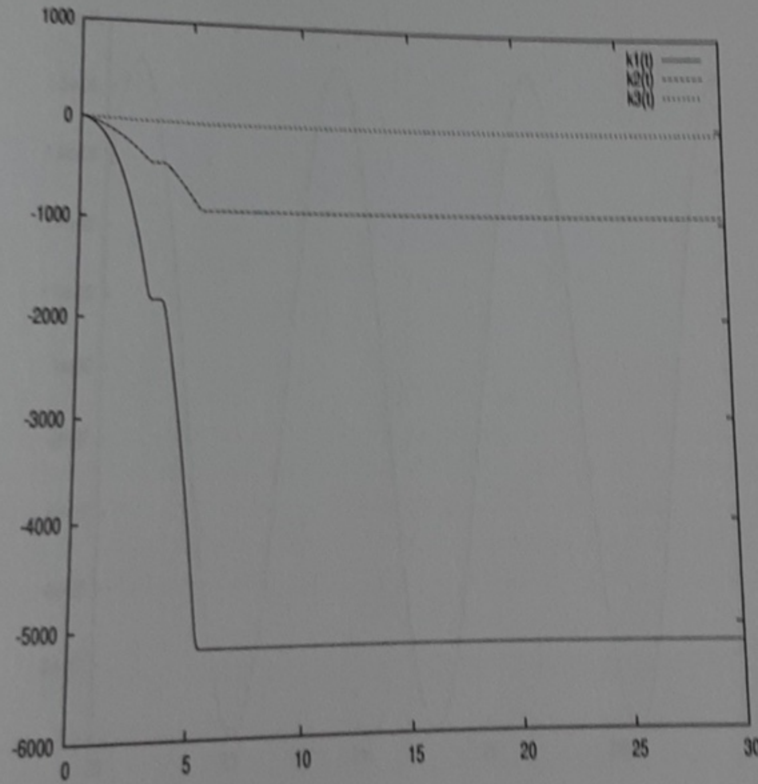

Figure 2.5.10: Histories of the feedback gains of the observer-like system: $k_1(t)$, $k_2(t)$, and $k_3(t)$.

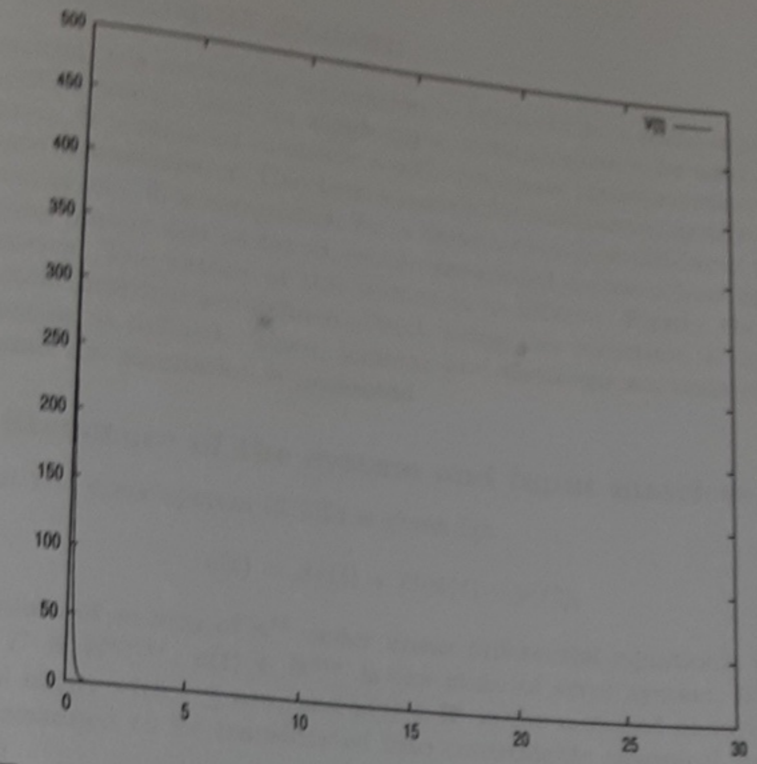

Figure 2.5.11: History of the Lyapunov-like function $V(t)$.

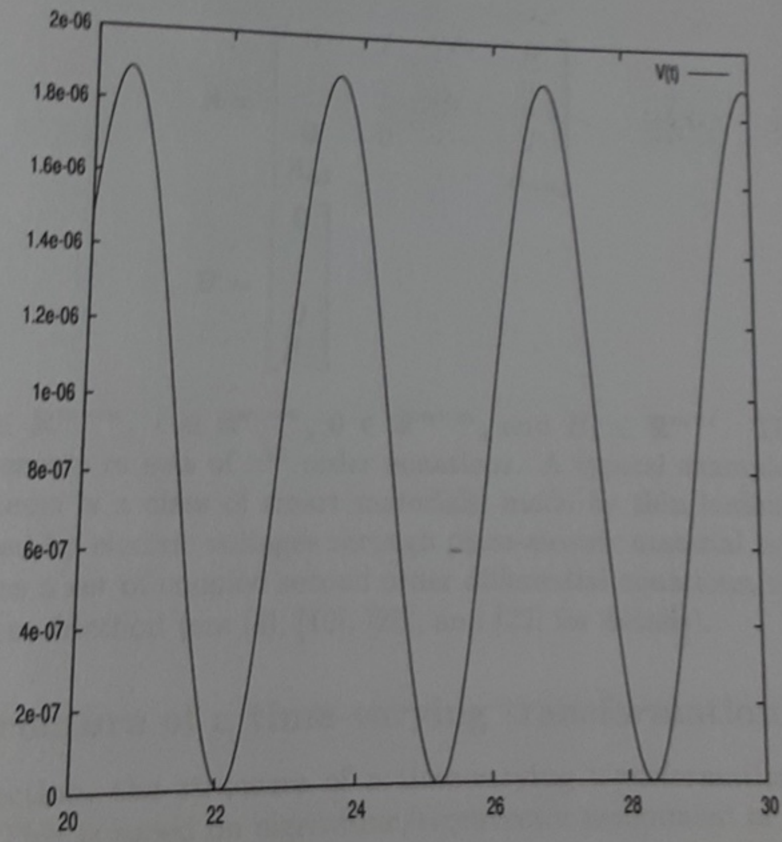

Figure 2.5.12: History of the Lyapunov-like function $V(t)$ at some later time interval.

2.6 Multi-input system

In this section, the method of estimation is extended to a multi-input system. Since the $T(t)$ matrix used for single-input system, cannot be used for multi-input system, it is required to obtain a different linear transformation which has appropriate characteristics. Can such a matrix for multi-input system be found? The answer is yes. It is shown that for a certain class of multi-input systems, a time-varying matrix can be found, whose associated matrices have appropriate characteristics. The outline of this section is as follows. Firstly, the structure of the system matrices are defined. Next, using this structure, a time-varying transformation is defined. Then, lemmas and theorems are constructed and, finally, numerical simulation is presented.

2.6.1 Structure of the system and input matrices

Recall that the error system (2.3.2) is given by:

$$\dot{e}(t) = Ae(t) + B(\bar{u}(t) - p(t)),$$

which consists of m sets of n^{th} order linear differential equations, where $A \in \mathbb{R}^{nm\times nm}$, $B \in \mathbb{R}^{nm\times l}$, $e(t) \in \mathbb{R}^{nm}$ is the state of error system, $\bar{u}(t) \in \mathbb{R}^{l}$ is the control input with $l \leq nm$, and $p(t) \in \mathbb{R}^{l}$ is the matched disturbance. The system is assumed to be transformed into controllable canonical form (see [7] for details).

In this investigation, the following structure of the system and input matrices are considered. Let

$$A = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ A_{n1} & \cdots & \cdots & A_{nn} \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B_1 \end{bmatrix},$$

where $A_{ij} \in \mathbb{R}^{m\times m}$, $I \in \mathbb{R}^{m\times m}$, $0 \in \mathbb{R}^{m\times m}$, and $B_1 \in \mathbb{R}^{m\times l}$. This type of system represents m sets of n$^{\text{th}}$ order equations. A typical example of such a class of systems is a class of smart materials, made by thin laminated plates and controlled by electric voltages through piezo-electric material layers, which is modelled as a set of coupled second order differential equations, utilising the Finite Element Method (see [4], [10], [20], and [22] for details).

2.6.2 Structure of a time-varying transformation matrix

In this subsection, the structure of a time-varying transformation matrix is determined. This is based on eigenvalue/eigenvector assignment of multi-input linear systems (see [26] and [28]).

Suppose there exists a matrix $K(t) \in \mathbb{R}^{l\times nm}$ such that $A + BK(t)$ is a diagonisable matrix (see Assumption 5), namely there exists a non-singular matrix $T(t)$ and a diagonal matrix $\Lambda(t)$ such that

$$T^{-1}(t)(A + BK(t))T(t) = \Lambda(t), \tag{2.6.1}$$

where $K(t) = [K_1(t)\ K_2(t)\ \cdots\ K_n(t)]$, $K_i(t) \in \mathbb{R}^{l\times m}$, $T(t) \in \mathbb{R}^{nm\times nm}$, and

$$T(t) = \begin{bmatrix} T_{11}(t) & \cdots & T_{1n}(t) \\ \vdots & \ddots & \vdots \\ T_{n1}(t) & \cdots & T_{nn}(t) \end{bmatrix}$$

with $T_{ij}(t) \in \mathbb{R}^{m\times m}$. It follows from (2.6.1) that

$$\bar{A}T(t) = T\Lambda(t)$$
$$\bar{A}(t) := A + BK(t) \tag{2.6.2}$$

Now, consider (2.6.2),

$$\bar{A}T(t) = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ \bar{A}_{n1}(t) & \cdots & \cdots & \bar{A}_{nn}(t) \end{bmatrix} \begin{bmatrix} T_{11}(t) & \cdots & T_{1n}(t) \\ \vdots & \ddots & \vdots \\ T_{n1}(t) & \cdots & T_{nn}(t) \end{bmatrix}$$
$$= \begin{bmatrix} T_{21}(t) & \cdots & T_{2n}(t) \\ \vdots & \ddots & \vdots \\ T_{n1}(t) & \cdots & T_{nn}(t) \\ \bar{T}_1(t) & \cdots & \bar{T}_n(t) \end{bmatrix}, \tag{2.6.3}$$

where $\bar{A}_{nj}(t) = A_{nj} + B_1K_j(t)$, $T_{ij}(t) \in \mathbb{R}^{m\times m}$ and $\bar{T}_j(t) = \bar{A}_{n1}T_{1j} + \bar{A}_{n2}T_{2j} + \cdots + \bar{A}_{nn}T_{nj}$. Also,

$$T(t)\Lambda(t) = \begin{bmatrix} T_{11}(t) & \cdots & T_{1n}(t) \\ \vdots & \ddots & \vdots \\ T_{n1}(t) & \cdots & T_{nn}(t) \end{bmatrix} \begin{bmatrix} \Lambda_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Lambda_n(t) \end{bmatrix}$$
$$= \begin{bmatrix} T_{11}(t)\Lambda_1(t) & \cdots & T_{1n}(t)\Lambda_n(t) \\ \vdots & \ddots & \vdots \\ T_{n1}(t)\Lambda_1(t) & \cdots & T_{nn}(t)\Lambda_n(t) \end{bmatrix}, \tag{2.6.4}$$

where $\Lambda(t) = diag(\Lambda_1(t), \Lambda_2(t), \cdots, \Lambda_n(t))$ and $\Lambda_j(t) \in \mathbb{R}^{m\times m}$. Substituting (2.6.3) and (2.6.4) in (2.6.2) gives

$$\begin{bmatrix} T_{21}(t) & \cdots & T_{2n}(t) \\ \vdots & \ddots & \vdots \\ T_{n1}(t) & \cdots & T_{nn}(t) \\ \bar{T}_1(t) & \cdots & \bar{T}_n(t) \end{bmatrix} = \begin{bmatrix} T_{11}(t)\Lambda_1(t) & \cdots & T_{1n}(t)\Lambda_n(t) \\ \vdots & \ddots & \vdots \\ T_{(n-1)1}(t)\Lambda_1(t) & \cdots & T_{(n-1)n}(t)\Lambda_n(t) \\ T_{n1}(t)\Lambda_1(t) & \cdots & T_{nn}(t)\Lambda_n(t) \end{bmatrix}$$

Therefore, for given $[T_{11}(t)\cdots T_{1n}(t)]$, if $\Lambda(t)$ is known then the remaining elements of the $T(t)$ can be determined and, hence, $T(t)$ is given by

$$T(t) = \begin{bmatrix} T_{11}(t) & \cdots & T_{1n}(t) \\ T_{11}(t)\Lambda_1(t) & \cdots & T_{1n}\Lambda_n(t) \\ \vdots & \ddots & \vdots \\ T_{11}(t)\Lambda_1^{n-1}(t) & \cdots & T_{1n}(t)\Lambda_n^{n-1}(t) \end{bmatrix}. \tag{2.6.5}$$

It is noticed that this matrix has a similar structure to the Vandermonde matrix which is used as the $T(t)$ matrix for a n^{th}order single-input system (see (2.5.4)). The basic philosophy for the analysis is the same as for the n^{th}order single-input system.

Remark 24 In this investigation, the matrices $T_{11}(t) \cdots T_{1n}(t)$ are determined as constant matrices.

2.6.3 Adaptive algorithm

An adaptive algorithm for the multi-input system is presented in this subsection. This adaptive algorithm for generating $\lambda_i(t)$ is the same as for the n^{th}order single-input system.

Algorithm 3 *One of the eigenvalues of the error system for the multi-input system, denoted by $\lambda_1(t)$, is determined as follows. Suppose δ and ϵ_e are specified constants, which are determined by control designer. At $t = t_0$, $\lambda_1(t) := -\lambda_0$ and $\dot{\lambda}_1(t) := -\lambda_{d0}$, where $\lambda_0 \in \mathbb{R}^+$ and $\lambda_{d0} := 0$ or δ are chosen by the control designer. Define $V(t) := \|\bar{e}(t)\|^2$. The structure of $\dot{\lambda}_1(t)$ is determined as follows:*

1. *let $\tau := t$ $(t \geq t_0)$;*
2. *evaluate $\bar{e}(\tau)$ and, hence, $V(\tau)$ is also obtained;*
3. *(a) if $(V(\tau) \leq \epsilon_e^2$ and $\dot{\lambda}_1(\tau) = -\delta)$ then $\tau_1 = \tau$ and $\dot{\lambda}_1(\cdot)$ has the following structure: $\dot{\lambda}_1(s) = f(s, \tau_1)$ for $s \geq \tau_1$;*
   (b) or if $(V(\tau) > \epsilon_e^2$ and $\dot{\lambda}_1(\tau) = 0)$ then $\tau_1 = \tau$ and $\dot{\lambda}_1(\cdot)$ has the structure: $\dot{\lambda}_1(s) = g(s, \tau_1)$ for $s \geq \tau_1$;
   (c) otherwise, the structure of $\dot{\lambda}_1(\cdot)$ is not changed;
4. *$t = t + \Delta t$ where Δt is a prescribed positive constant;*
5. *evaluate $\dot{\lambda}_1(t)$ using the given structure of $\dot{\lambda}_1(s)$;*

where the functions $t \mapsto f(t,\tau)$ and $t \mapsto g(t,\tau)$ are defined as follows:

$$f(t,\tau) := \begin{cases} \frac{1}{2}\delta \sin(\omega(t-\tau) + \frac{3}{2}\pi) - \frac{1}{2}\delta, & \tau \leq t \leq \frac{\pi}{\omega} + \tau \\ 0, & \frac{\pi}{\omega} + \tau < t \end{cases}$$

$$g(t,\tau) := \begin{cases} \frac{1}{2}\delta \sin(\omega(t-\tau) + \frac{1}{2}\pi) - \frac{1}{2}\delta, & \tau \leq t \leq \frac{\pi}{\omega} + \tau \\ -\delta, & \frac{\pi}{\omega} + \tau < t \end{cases}$$

where ω is specified constant.

The remaining eigenvalues of the multi-input error system, which are $\lambda_2(t) \cdots \lambda_{nm}(t)$, are determined as follows:

$$\lambda_i(t) = \kappa_i \lambda_1(t),$$

where κ_i $(i = 2 \cdots nm)$ are prescribed positive constants determined by control designer, κ_i satisfy $\kappa_i \neq 1$ for all i, with $\kappa_i \neq \kappa_j$ for $i \neq j$.

Remark 25 Since, $T(t)$, $\Lambda(t)$, and A are known matrices, and (A, B) is controllable, the feedback gains to the observer-like system (2.2.4) can be determined using

$$BK(t) = T(t)\Lambda(t)T^{-1}(t) - A. \tag{2.6.6}$$

Remark 26 Using $\lambda_i(t)$, the time-varying matrices $\Lambda_j(t)$ are defined by:

$$\Lambda_j(t) = diag(\lambda_{(j-1)m+1}(t), \cdots, \lambda_{jm}(t)).$$

2.6.4 Main result

In this subsection, appropriate lemmas and theorems are constructed. In particular, using the matrix $T(t)$ defined in Subsection 2.6.2, it is shown that $||T^{-1}(t)B||_1$ and $||T^{-1}(t)\dot{T}(t)||_1$, have appropriate characteristics; that is they are non-increasing functions.

Lemma 17 *For given constant matrices $T_{11}, T_{12}, \cdots, T_{1n}$, where $T_{1j} \in \mathbb{R}^{m\times m}$, if $T_{1i} = T_{11}$ for $i = 2, \cdots n$, and if the row vectors of the matrix $[T_{11},\ T_{12},\ \cdots,\ T_{1n}]$ are linearly independent, and if $\lambda_i(t) \neq \lambda_j(t)$ $(i \neq j)$, then the $T(t)$ matrix given by (2.6.5) is non-singular.*

Proof Since, $T_{1i} = T_{11}$ for $i = 2, \cdots, n$, (2.6.5) can be represented as follows:

$$T(t) = \begin{bmatrix} t_{11} & \cdots & t_{1m} & \cdots & t_{11} & \cdots & t_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ t_{m1} & \cdots & t_{mm} & \cdots & t_{m1} & \cdots & t_{mm} \\ \vdots & & \vdots & & \vdots & & \vdots \\ t_{11}\lambda_1^{i-1}(t) & \cdots & t_{1m}\lambda_m^{i-1}(t) & \cdots & t_{11}\lambda_{(n-1)m+1}^{i-1}(t) & \cdots & t_{1m}\lambda_{nm}^{i-1}(t) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ t_{m1}\lambda_1^{i-1}(t) & \cdots & t_{mm}\lambda_m^{i-1}(t) & \cdots & t_{m1}\lambda_{(n-1)m+1}^{i-1}(t) & \cdots & t_{mm}\lambda_{nm}^{i-1}(t) \\ \vdots & & \vdots & & \vdots & & \vdots \\ t_{11}\lambda_1^{n-1}(t) & \cdots & t_{1m}\lambda_m^{n-1}(t) & \cdots & t_{11}\lambda_{(n-1)m+1}^{n-1}(t) & \cdots & t_{1m}\lambda_{nm}^{n-1}(t) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ t_{m1}\lambda_1^{n-1}(t) & \cdots & t_{mm}\lambda_m^{n-1}(t) & \cdots & t_{m1}\lambda_{(n-1)m+1}^{n-1}(t) & \cdots & t_{mm}\lambda_{nm}^{n-1}(t) \end{bmatrix}. \tag{2.6.7}$$

Assume the first $m \times (i-1)$ $(1 < i \leq n)$ row vectors of (2.6.7) are linearly independent.

Under this condition, assume the first $m \times (i-1) + 1$ $(n \geq i > 1)$ row vectors of (2.6.7) are linearly dependent. Hence, there exist scalars α_l $(l =$

$1, \cdots, m \times (i-1)+1)$ such that following relation holds:

$$
\begin{aligned}
0 &= (\alpha_1 t_{11} + \cdots + \alpha_m t_{m1}) \\
&\quad + \cdots + \\
&\quad + [\alpha_{(i-2)m+1} t_{11} \lambda_k^{i-2}(t) + \cdots + \alpha_{(i-1)m} t_{m1} \lambda_k^{i-2}(t)] \\
&\quad + \alpha_{(i-1)m+1} t_{11} \lambda_k^{i-1}(t) \\
&= (\alpha_1 t_{11} + \cdots + \alpha_m t_{m1}) \\
&\quad + \cdots + \\
&\quad + (\alpha_{(i-2)m+1} t_{11} + \cdots + \alpha_{(i-1)m} t_{m1}) \lambda_k^{i-2}(t) \\
&\quad + \alpha_{(i-1)m+1} t_{11} \lambda_k^{i-1}(t), \quad k = 1,\ m+1,\ 2m+1,\ \cdots,\ (n-1)\times m+1.
\end{aligned} \tag{2.6.8}
$$

Consider the following polynominal.

$$f(x(t)) = \beta_1 + \cdots + \beta_{i-1} x^{i-2}(t) + \beta_i x^{i-1}(t), \tag{2.6.9}$$

where $\beta_1 = \alpha_1 t_{11} + \cdots + \alpha_m t_{m1}$, $\beta_{i-1} = \alpha_{(i-2)m+1} t_{11} + \cdots + \alpha_{(i-1)m} t_{m1}$, and $\beta_i = \alpha_{(i-1)m+1} t_{11}$. Then, the right hand side of (2.6.8) can be treated as the value of the polynominal (2.6.9) with $x(t) = \lambda_k(t)$ for each k. If the equation (2.6.8) is valid for all k, since $k = 1,\ m+1,\ 2m+1,\ \cdots,\ (n-1)\times m+1$, the equation $f(x(t)) = 0$, where $f(x(t))$ is given by (2.6.9), should have n distinct solutions $x(t) = \lambda_1(t),\ \lambda_{m+1}(t),\ \lambda_{2m+1}(t),\ \cdots,\ \lambda_{(n-1)\times m+1}(t)$. However, since the order of the polynominal $f(x(t))$ is $(i-1)$, it is not possible. Thus, the assumption, which is the first $m \times (i-1)+1$ row vectors of (2.6.7) are linearly dependent, is invalid. Thus, the first $m \times (i-1)+1$ row vectors of (2.6.7) are linearly independent.

It is clear that the same implication can be made when the first $m\times(i-1)+2$ to $m\times(i-1)$ row vectors of (2.6.7), where $1 < i \leq n$, are considered. Moreover, by definition, the first m row vectors of (2.6.7) are linearly independent. Thus, this argument can be repeated to show that all the row vectors of (2.6.7) are linearly independent and, therefore, $T(t)$, which is given by (2.6.7), is non-singular under the constraints given in the statement of this lemma. ∎

Lemma 18 *If Algorithm 3 is used,* $||\bar{t}_j(t)||$, *which is the* j^{th} *column vector of* $T^{-1}(t)$, *is a non-increasing function if* $j \geq m+1$ *and is constant if* $j = 1, \cdots, m$.

Proof It is firstly shown that the 'direction' of the i^{th} row vectors of $T(t)$ matrix is fixed for all time t. Let divide $T(t)$ as following submatrices:

$$T(t) = \begin{bmatrix} M_1^t(t) & M_2^t(t) & \cdots & M_n^t(t) \end{bmatrix}^t,$$

where $M_j(t) \in \mathbb{R}^{m\times m}$. Then, $M_p(t)$ is expressed as follows:

$$M_p(t) = \left[T_{11}\Lambda_1^{p-1}(t) \cdots T_{1n}\Lambda_n^{p-1}(t)\right]. \tag{2.6.10}$$

Let

$$T_{1j} = \begin{bmatrix} t_{11}^{1j} & t_{12}^{1j} & \cdots & t_{1m}^{1j} \\ t_{21}^{1j} & t_{22}^{1j} & \cdots & t_{2m}^{1j} \\ \vdots & \ddots & \ddots & \vdots \\ t_{m1}^{1j} & \cdots & \cdots & t_{mm}^{1j} \end{bmatrix}.$$

Thus,

$$\mathcal{T}_{1,j}\Lambda_j^{p-1}(t) = \begin{bmatrix} t_{11}^{1j}\lambda_{(j-1)m+1}^{p-1}(t) & \cdots & t_{1m}^{1j}\lambda_{jm}^{p-1}(t) \\ \vdots & \ddots & \vdots \\ t_{m1}^{1j}\lambda_{(j-1)m+1}^{p-1}(t) & \cdots & t_{mm}^{1j}\lambda_{jm}^{p-1}(t) \end{bmatrix}. \tag{2.6.11}$$

Let consider the vector $t_i(t)$, which is the i^{th} row vector of $M_p(t)$. As a consequence of (2.6.10) and (2.6.11),

$$t_i(t) = \left[t_{j1}^{11}\lambda_1^{p-1}(t)\ t_{j2}^{11}\lambda_2^{p-1}(t)\ \cdots\ t_{jm}^{11}\lambda_m^{p-1}(t)\ \cdots\ t_{j1}^{1n}\lambda_{(n-1)m+1}^{p-1}(t)\ \cdots\ t_{mm}^{1n}\lambda_{nm}^{p-1}(t)\right].$$

In view of Algorithm 3, $\lambda_i(t)$ $(i = 1 \cdots n)$ must satisfy the following relations:

$$\lambda_1(t) : \lambda_2(t) : \cdots : \lambda_n(t) = 1 : \kappa_2 : \cdots : \kappa_n$$

where κ_i are given constants. Hence, t_i has the form

$$t_i = \lambda_1^{p-1}(t)\left[\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{nm}\right],$$

where α_i are constants. Thus, the 'direction' of each $t_i(t)$ is fixed for all t. Note that since the index p is arbitary, this result implies that the 'direction' of any row vector of $T(t)$ is fixed for all t.

Next, the norm of a column vector of $T^{-1}(t)$ is investigated. Define the i^{th} row vector of $T(t)$ and l^{th} column vector of $T^{-1}(t)$ as $\tau_i(t)$ and $\bar{\tau}_l(t)$ respectively. Then, by definition,

$$T(t)T^{-1} = I$$

$$\begin{bmatrix} \tau_1(t) \\ \vdots \\ \tau_{nm}(t) \end{bmatrix} \begin{bmatrix} \bar{\tau}_1(t) & \cdots & \bar{\tau}_{nm}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 1 \end{bmatrix}.$$

Hence,

$$\tau_i(t) \cdot \bar{\tau}_i(t) = 1,$$
$$\tau_i(t) \cdot \bar{\tau}_j(t) = 0, \quad i \neq j.$$

As these equations indicate, $\bar{\tau}_l(t)$ and $\tau_i(t)$ $(l \neq i)$ is orthogonal for any time t. Moreover, it is already shown that the 'direction' of each $\tau_i(t)$ $(i = 1, \cdots, nm)$ is fixed for all t. Hence, by Lemma 11, the 'direction' of each $\bar{\tau}_l(t)$ $(l = 1, \cdots, nm)$ is unique and the 'direction' of each $\bar{\tau}_l(t)$ is fixed for all t. Since the 'directions' of each $\tau_l(t)$ and $\bar{\tau}_l(t)$ are fixed, by Lemma 12, the magnitude of each $\frac{\tau_l(t)\cdot\bar{\tau}_l(t)}{||\tau_l(t)||\,||\bar{\tau}_l(t)||}$ $(l = 1, \cdots nm)$ is constant for all t. Thus, there exists a constant α_l such that

$$||\bar{\tau}_l(t)|| = \alpha_l \frac{1}{||\tau_l(t)||}.$$

Recall that

$$||\tau_l(t)|| = \lambda_1^{p-1}(t)\sqrt{\gamma_1^2 + \cdots + \gamma_{nm}^2}$$
$$= \lambda_1^{p-1}(t)\bar{\gamma},$$

where $\bar{\gamma}$ is a positive constant. Therefore,

$$||\bar{\tau}_l(t)|| = \frac{\alpha_l}{\bar{\gamma}} \frac{1}{\lambda_1^{p-1}(t)}.$$

If $l = 1 \cdots m$, then $p - 1 = 0$, and if $l \geq m + 1$, then $p - 1 \geq 1$. Therefore, if Algorithm 3 is used, $||\bar{\tau}_l(t)||$ is a non-increasing function when $l \geq m + 1$ and is constant if $l = 1 \cdots m$. ■

Lemma 19 *If Algorithm3 is used, $||T^{-1}(t)B||_1$ is non-increasing function when $n > 1$.*

Proof Consider the j^{th} column vector of $T^{-1}(t)$. By Lemma 18, its norm is given by

$$||\bar{t}_j(t)|| = \alpha_j \frac{1}{\lambda_1^{p-1}(t)},$$

where α_j is positive constant. Therefore, $\bar{t}_j(t)$ has the form:

$$\bar{t}_j^t(t) = \left[\xi_{1j}(t)\frac{1}{\lambda_1^{p-1}(t)} \quad \cdots \quad \xi_{nm,j}(t)\frac{1}{\lambda_1^{p-1}(t)}\right],$$

where $|\xi_{ij}(t)| \leq \alpha_j$ for all t. Hence, $T^{-1}(t)$ can be expressed as

$$T^{-1}(t) = \begin{bmatrix} C_{11}(t) & C_{12}(t)\frac{1}{\lambda_1(t)} & \cdots & C_{1n}(t)\frac{1}{\lambda_1^{n-1}(t)} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1}(t) & C_{n2}(t)\frac{1}{\lambda_1(t)} & \cdots & C_{nn}(t)\frac{1}{\lambda_1^{n-1}(t)} \end{bmatrix}, \qquad (2.6.12)$$

where $C_{ij}(t) \in \mathbb{R}^{m\times m}$ are bounded by constants. Thus,

$$T^{-1}(t)B = \begin{bmatrix} C_{11}(t) & C_{12}(t)\frac{1}{\lambda_1(t)} & \cdots & C_{1n}(t)\frac{1}{\lambda_1^{n-1}(t)} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1}(t) & C_{n2}(t)\frac{1}{\lambda_1(t)} & \cdots & C_{nn}(t)\frac{1}{\lambda_1^{n-1}(t)} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B_1 \end{bmatrix}$$

$$= \frac{1}{\lambda_1^{n-1}(t)} \begin{bmatrix} C_{1n}(t)B_1 \\ \vdots \\ C_{nn}(t)B_1 \end{bmatrix}.$$

Thus, since $C_{ij}(t)$ are bounded by constants, there exists a positive constant γ such that

$$||T^{-1}(t)B||_1 = \gamma \frac{1}{|\lambda_1^{n-1}(t)|}. \qquad (2.6.13)$$

Therefore, $||T^{-1}(t)B||_1$ is non-increasing function if Algorithm 3 is used and when $n > 1$. ■

Lemma 20 *If Algorithm 3 is used, $||T^{-1}(t)\dot{T}(t)||_1$ is a non-increasing function.*

Proof Recall that

$$T(t) = \begin{bmatrix} T_{11} & \cdots & T_{1n} \\ T_{11}\Lambda_1(t) & \cdots & T_{1n}\Lambda_n(t) \\ \vdots & \ddots & \vdots \\ T_{11}\Lambda_1^{n-1}(t) & \cdots & T_{1n}\Lambda_n^{n-1}(t) \end{bmatrix},$$

where $T_{ij} \in \mathbb{R}^{m\times m}$ and $\Lambda_i(t) \in \mathbb{R}^{m\times m}$. Hence,

$$\dot{T}(t) = \begin{bmatrix} 0 & \cdots & 0 \\ T_{11}\dot{\Lambda}_1(t) & \cdots & T_{1n}\dot{\Lambda}_n(t) \\ \vdots & \ddots & \vdots \\ (n-1)T_{11}\dot{\Lambda}_1(t)\Lambda_1^{n-2}(t) & \cdots & (n-1)T_{1n}\dot{\Lambda}_n(t)\Lambda_n^{n-2}(t) \end{bmatrix}.$$

By definition, $\Lambda_j^{i-2}(t)$ are expressed as follows:

$$\Lambda_j^{i-2}(t) = \begin{bmatrix} \lambda_k^{i-2}(t) & & \\ & \ddots & \\ & & \lambda_l^{i-2}(t) \end{bmatrix}.$$

As a consequence of Algorithm 3,

$$\Lambda_j^{i-2}(t) = \lambda_1^{i-2}(t) \begin{bmatrix} \kappa_k^{n-2} & & \\ & \ddots & \\ & & \kappa_l^{n-2} \end{bmatrix},$$

where κ_i are specified constants. Hence, there exist matrices $\tilde{T}_{ij}(t)$ such that

$$\dot{T}(t) = \begin{bmatrix} 0 & \cdots & 0 \\ \tilde{T}_{21}(t) & \cdots & \tilde{T}_{2n}(t) \\ \vdots & \ddots & \vdots \\ (n-1)\lambda_1^{n-2}(t)\tilde{T}_{n1}(t) & \cdots & (n-1)\lambda_1^{n-2}(t)\tilde{T}_{nn}(t) \end{bmatrix}.$$

Let $a_i(t)$ denote the i^{th} sub-matrix of $T^{-1}(t)$, given in (2.6.12), which consists of the $\{(i-1)m+1\}^{\text{th}}$ to $\{i \times m\}^{\text{th}}$ row vectors of $T^{-1}(t)$, that is

$$a_i(t) := \begin{bmatrix} C_{i1}(t) & C_{i2}(t)\frac{1}{\lambda_1(t)} & \cdots & C_{in}(t)\frac{1}{\lambda_1^{n-1}(t)} \end{bmatrix}.$$

Similarly, let $b_j(t)$ denote the j^{th} sub-matrix of $\dot{T}(t)$, which consists of the $\{(j-1)m+1\}^{\text{th}}$ to $\{j \times m\}^{\text{th}}$ column vectors of $\dot{T}(t)$, namely,

$$b_j(t) := \begin{bmatrix} 0 \\ \tilde{T}_{2j}(t) \\ \vdots \\ (n-1)\lambda_1^{n-2}(t)\tilde{T}_{nj}(t) \end{bmatrix}.$$

Hence,

$$\begin{aligned} a_i b_j &= C_{i1}(t)\cdot 0 + \frac{1}{\lambda_1(t)}C_{i2}(t)\tilde{T}_{2j}(t) + \cdots + (n-1)\lambda_1^{n-2}\frac{1}{\lambda_1^{n-1}}C_{in}(t)\tilde{T}_{nj}(t) \\ &= \frac{1}{\lambda_1(t)}C_{i2}(t)\tilde{T}_{2j}(t) + \cdots + (n-1)\frac{1}{\lambda_1(t)}C_{in}(t)\tilde{T}_{nj}(t). \end{aligned} \tag{2.6.14}$$

Since $C_{ij}(t)$ are bounded by constants and $\tilde{T}_{ij}(t)$ are bounded, $a_i(t)b_j(t)$ are decreasing function. Hence, $T^{-1}(t)\dot{T}(t)$ is a non-increaing function and so the result follows. ■

Remark 27 Using the results of Lemma 19 and 20, all the analysis for second order single-input systems can be applied to multi-input systems, because $||T^{-1}(t)B||_1$ and $||T^{-1}(t)\dot{T}(t)||_1$ have the required characteristics. See Remark 21 for a detailed explanation.

Remark 28 In view of Remarks 27 and 21, $\epsilon(t)$ for the multi-input system is given by

$$\epsilon(t) = -\frac{\alpha||T^{-1}(t)B||_1}{\sigma_{\max}(\Lambda(t)) + ||T^{-1}(t)\dot{T}(t)||_1}. \tag{2.6.15}$$

Theorem 5 *For the multi-input system (2.2.5), $\bar{u}(t) \approx p(t)$ can be achieved for t sufficiently large, where $p(t)$ is the matched disturbance/uncertainty in system (2.2.5) and $\bar{u}(t)$ is the control input to the observer-like system (2.2.4), using observer-like system (2.2.4), feedforward filter (2.2.8), modified reference signal (2.2.9) and the feedback control defined by (2.6.6) and Algorithm3.*

Proof In the view of Remark 27, the proof is an immediate consequence of Lemmas 19 and 20. ■

Remark 29 As well as previous class of systems, if the opposite sign of the 'estimate' of the matched disturbance, which is $-\bar{u}(t) \approx -p(t)$, is fed back to system (2.2.5), the matched disturbance in system (2.2.5) will be cancelled out.

2.6.5 Simulation example

In this subsection, a numerical simulation is presented to demonstrate the method developed for a multi-input system. It will be recognized that, as for single-input systems, the method proposed is simple, but effective.

Configuration

Consider a coupled second-order multi-input system given as follows:

$$\dot{r}(t) = Ar(t) + B(u(t) + d(t)),$$

where $r(t) \in \mathbb{R}^4$ is the state of the system, $u(t) \in \mathbb{R}^2$ is the control input, and $d(t) \in \mathbb{R}^2$ is a disturbance. The system and input matrices are given as follows:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & -0.1 & -5 & -0.05 \\ -0.2 & -20 & -0.1 & -4.0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For simulation purposes, the disturbance $d(t) = [d_1(t)\ d_2(t)]^t$ is chosen as:

$$d_1(t) = 0.5\sin t + 2.0r_1(t) + 0.03r_2(t) + 1.0r_3(t) + 0.01r_4(t)$$
$$d_2(t) = 0.3\sin t + 0.01r_1(t) + 3r_2(t) + 0.01r_3(t) + 3r_4(t)$$

The initial condition is $x(t_0) = [3\ 2\ 0\ 0]^t$. Also, for adaptive algorithm Algorithm 3, the parameters are chosen to be: $\delta = 2.0$, $\kappa_2 = 2.0$, $\kappa_3 = 3.0$, $\kappa_4 = 4.0$,

$\epsilon_e^2 = 4.0 \times 10^{-4}$, $\omega = 10$ $\lambda_1(t_0) = -2.0$, and $\dot{\lambda}_1(t_0) = 0$. Also, $T(t)$ matrix is chosen as follows:

$$T(t) = \begin{bmatrix} T_{11} & T_{12} \\ T_{11}\Lambda_1(t) & T_{12}\Lambda_2(t) \end{bmatrix}, \tag{2.6.16}$$

where $\Lambda_1(t) = diag(\lambda_1(t), \lambda_2(t))$, $\Lambda_2(t) = diag(\lambda_3(t), \lambda_4(t))$,

$$T_{11} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{2.6.17}$$

and

$$T_{12} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2.6.18}$$

In addition, estimated disturbances are used to cancel out effect of disturbance to the system; i.e. the opposite sign of the estimated disturbances are fed back to the system. The simulation program has been implemented using the following environment/algorithms:

Programing language: C++;

Compiler: g++ version 3.2.2;

Algorithm to obtain numerical solution of ODE: Runge-Kutta (see [31]);

Time step for Runge-Kutta algorithm: 1.0×10^{-5};

Algorithm to obtain inverse matrix: Gauss-Jordan Elimination (see [31]).

Simulation results

The open-loop response of the system, with disturbances, is shown in Figures 2.6.1, 2.6.2, 2.6.3, and 2.6.4. It is clear that in the presence of disturbances, the states of the system are perturbed around the equilibrium.

The closed-loop response of the system is shown in Figures 2.6.5, 2.6.6, 2.6.7, and 2.6.8. Unlike the open-loop system, these states converge to their equilibrium. The states $r_1(t)$ and $r_2(t)$, at later time, are illustrated in Figures 2.6.9 and 2.6.10. In those figures, it is observed that the states do not converge to the equilibrium exactly; in particular, it appears that the amplitude of the oscillation in $r_1(t)$ is becoming larger, but it remains within some small interval about the origin. The actual and estimated disturbances are shown in Figures 2.6.11 and 2.6.12. For both figures, solid line represents the actual disturbance, and the dashed line represents the estimated disturbance. In both figures, it is observed that the estimated disturbances converge to actual disturbances very rapidly. Difference between the actual and estimated disturbances at some later time intervals are shown in Figures 2.6.13 and 2.6.14. For both figures, it is observed that estimation errors are very small and, therefore, the estimated disturbances are nearly the same as the actual disturbances.

The histories of the eigenvalues of the error system and the feedback gains of the observer-like system are shown at Figures 2.6.15, 2.6.16, and 2.6.17. In those figures, it is observed that each values are increased or decreased until they reach threshold values and after reaching to these values, they remain there. The history of the Lyapunov-like function is shown in Figure 2.6.18. It is observed that the value of this function decreases rapidly.

at later time, is shown in Figure 2.6.19. In this figure, it is observed that this function converges to and remains within prescribed limits; i.e. $V \leq 4.0 \times 10^{-4}$ for sufficiently large time. Therefore, the simulation confirms the results of the analysis.

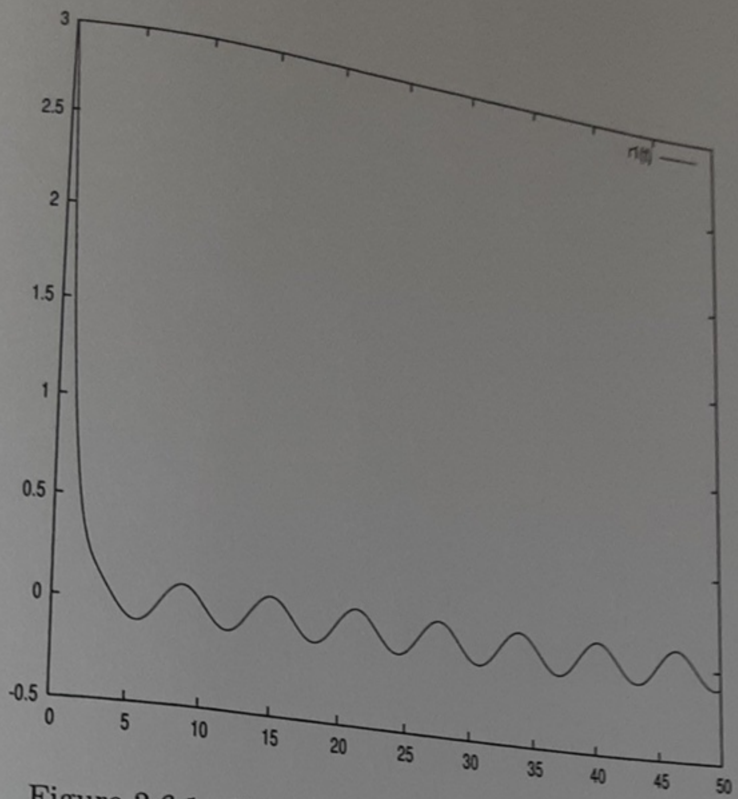

Figure 2.6.1: Open-loop response of the state $r_1(t)$.

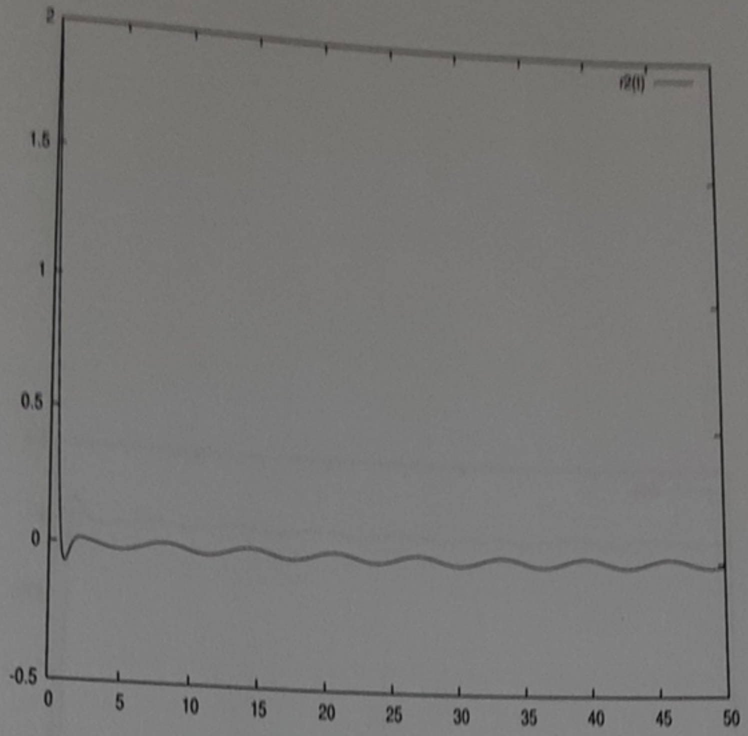

Figure 2.6.2: Open-loop response of the state $r_2(t)$.

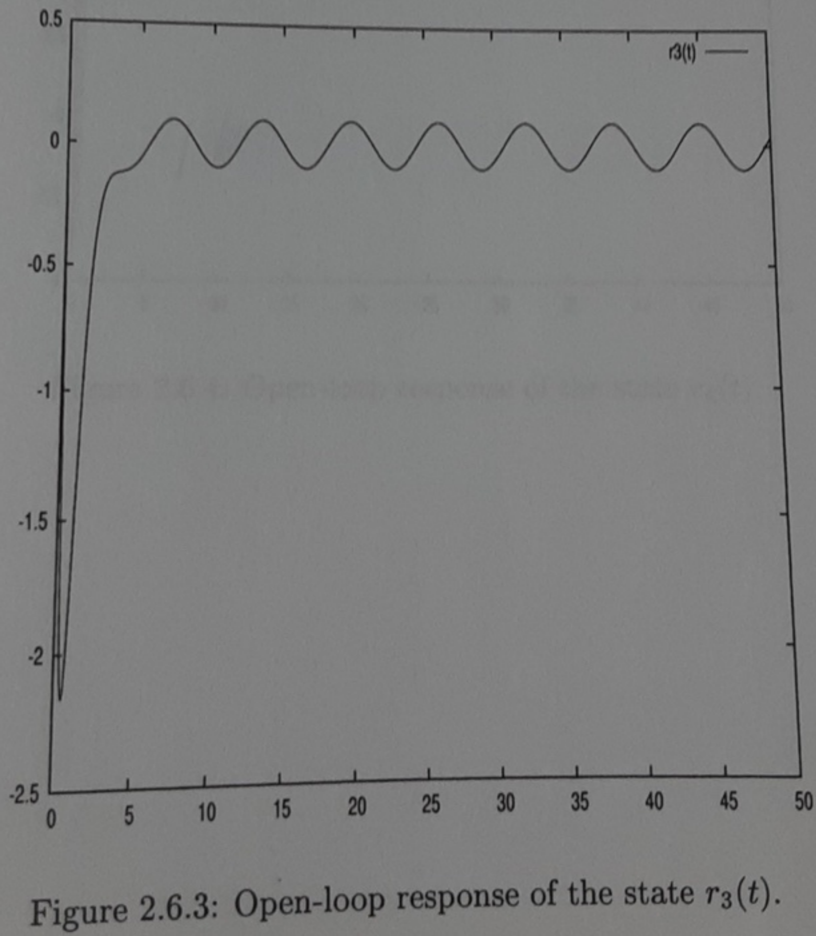

Figure 2.6.3: Open-loop response of the state $r_3(t)$.

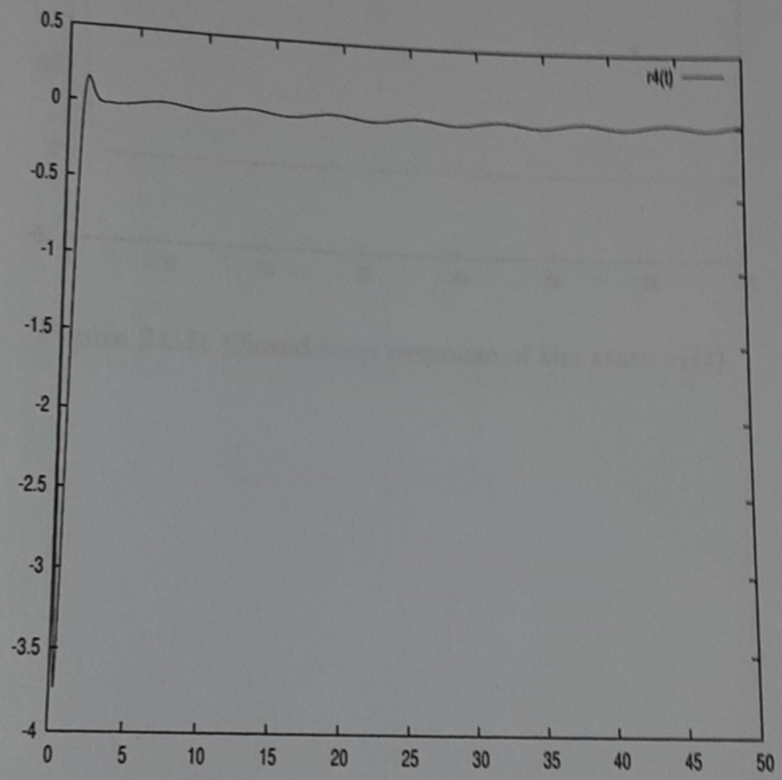

Figure 2.6.4: Open-loop response of the state $r_4(t)$.

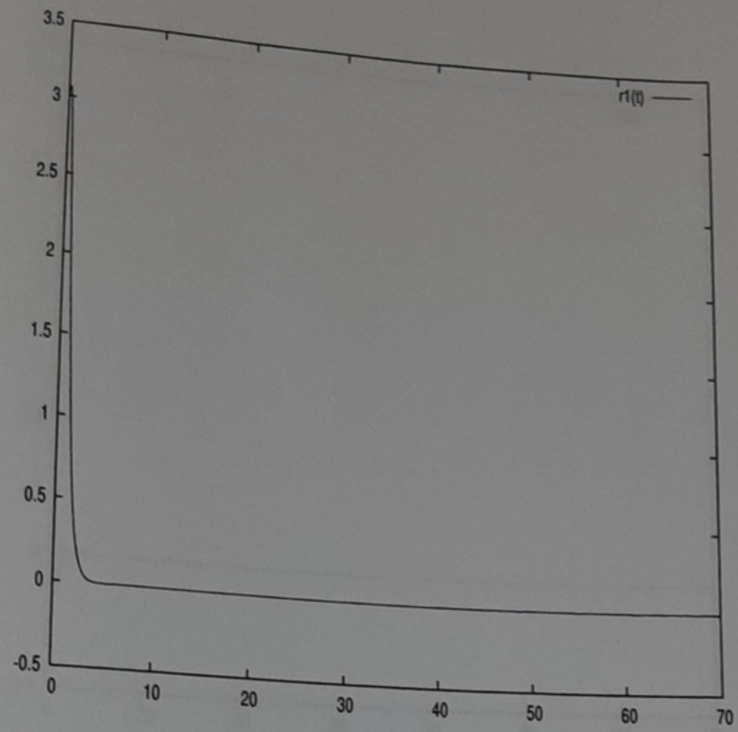

Figure 2.6.5: Closed-loop response of the state $r_1(t)$.

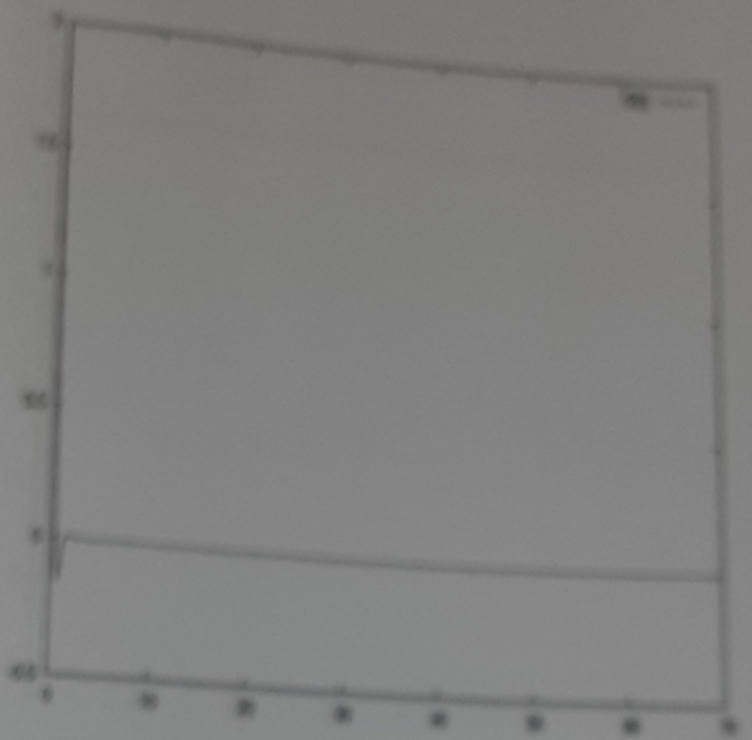

Figure 2.6.6: Closed-loop response of the state $x_2(t)$.

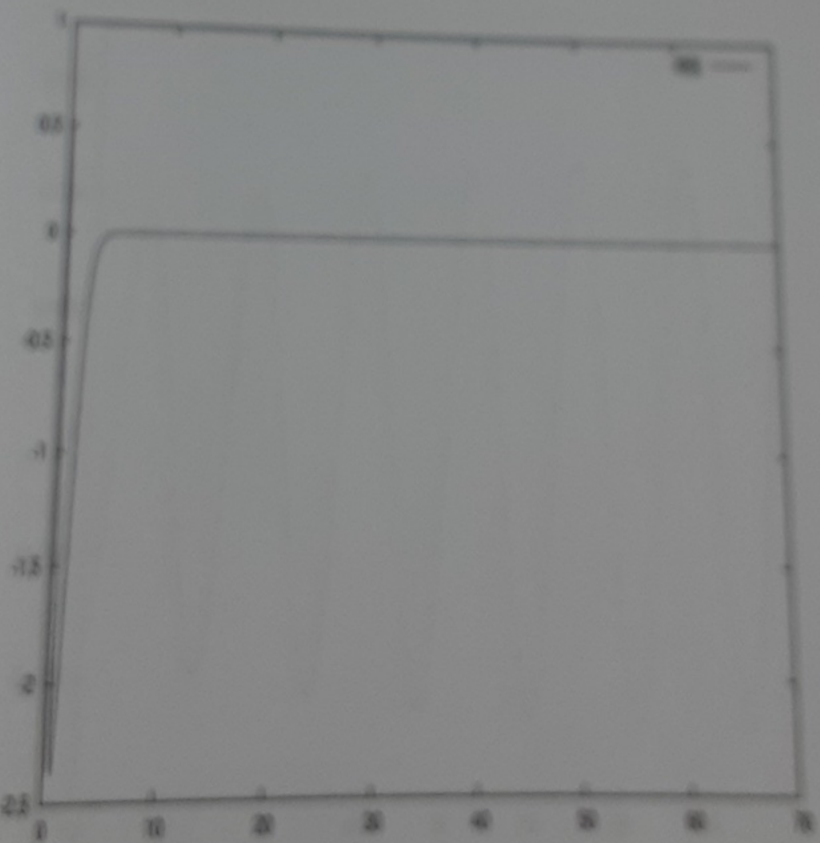

Figure 2.6.7: Closed-loop response of the state $x_1(t)$.

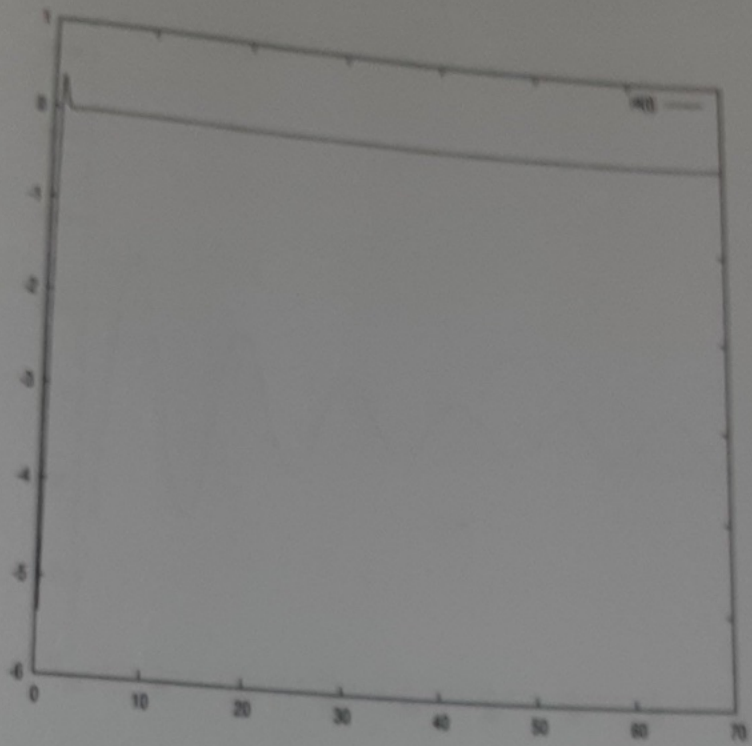

Figure 2.6.8: Closed-loop response of the state $r_4(t)$.

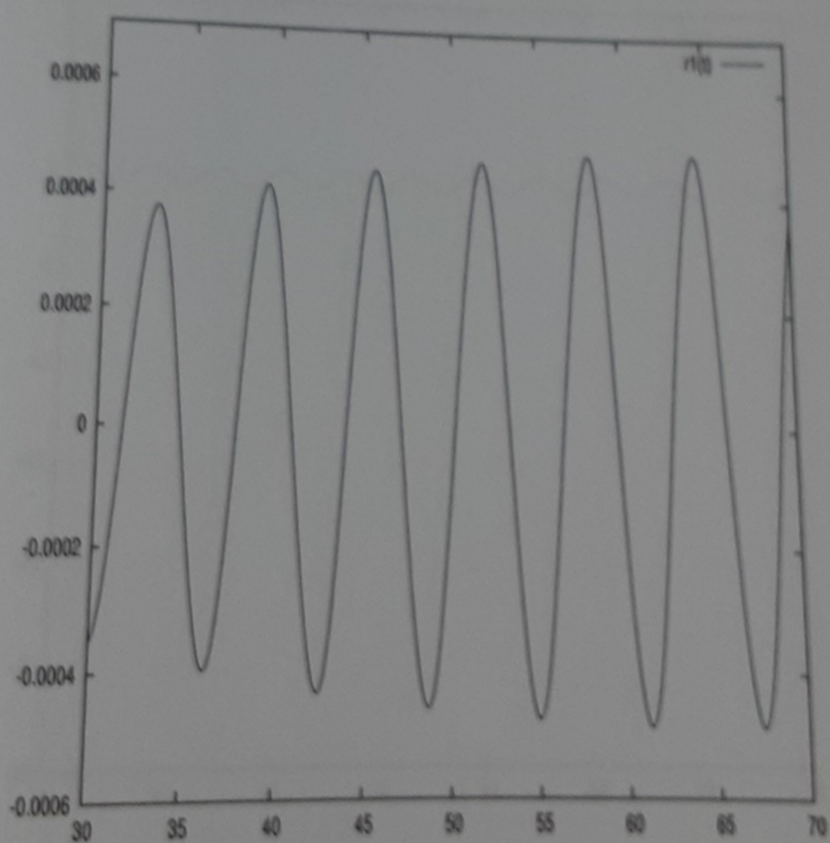

Figure 2.6.9: Closed-loop response of the state $r_1(t)$.

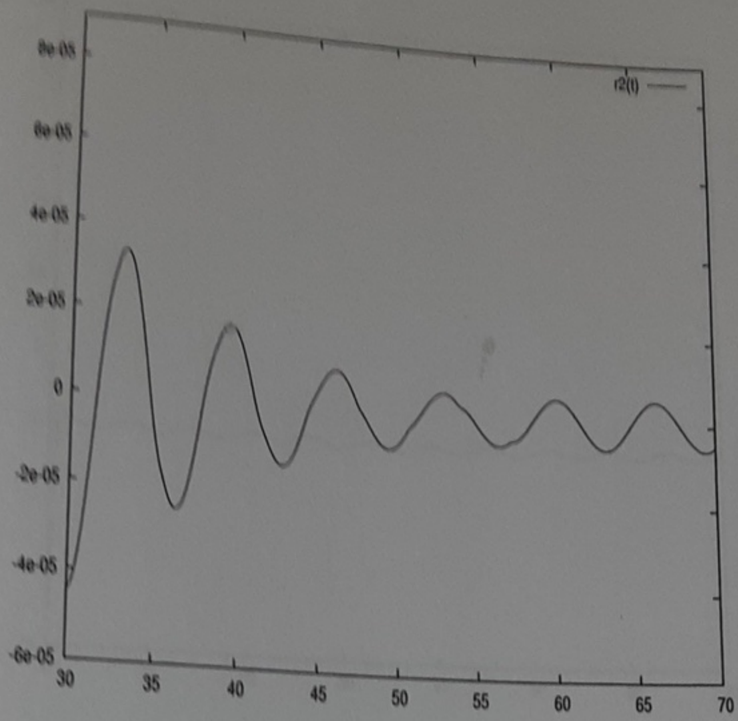

Figure 2.6.10: Closed-loop response of the state $r_2(t)$.

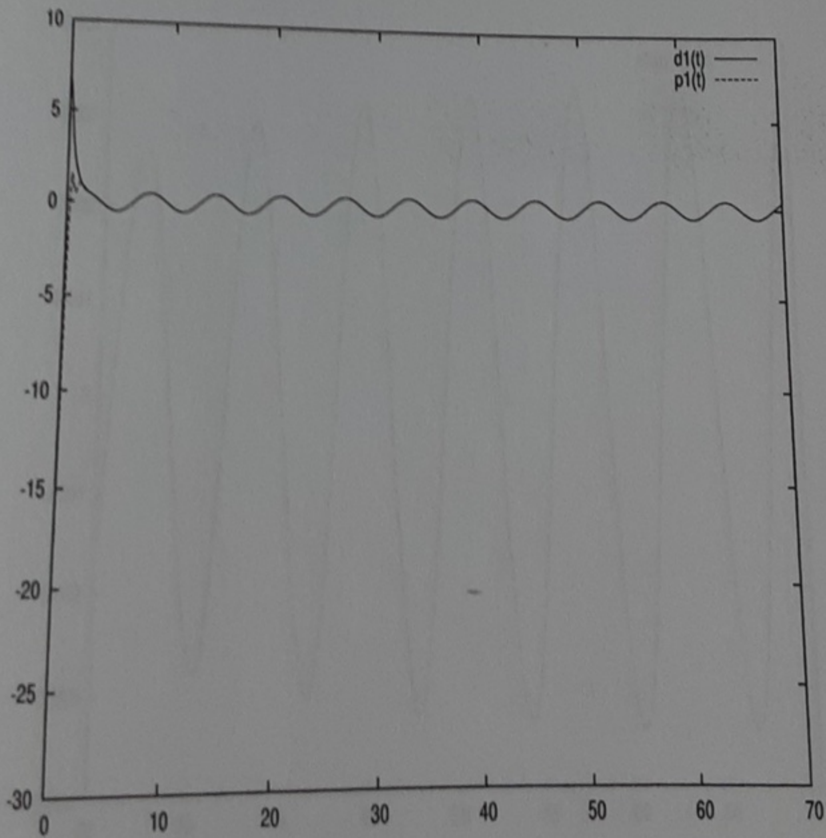

Figure 2.6.11: The actual and estimated disturbance, $d_1(t)$ and $p_1(t)$, respectively.

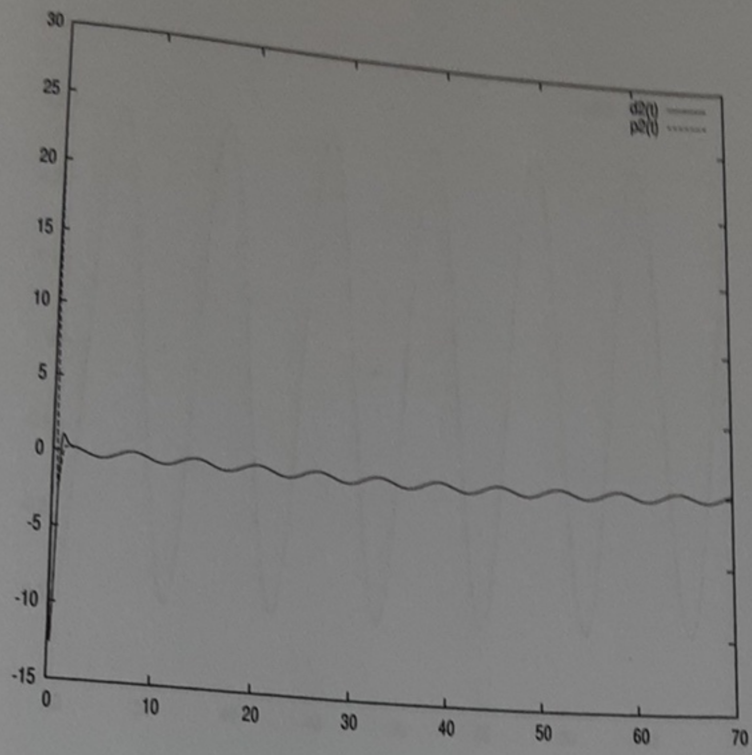

Figure 2.6.12: The actual and estimated disturbance, $d_2(t)$ and $p_2(t)$, respectively.

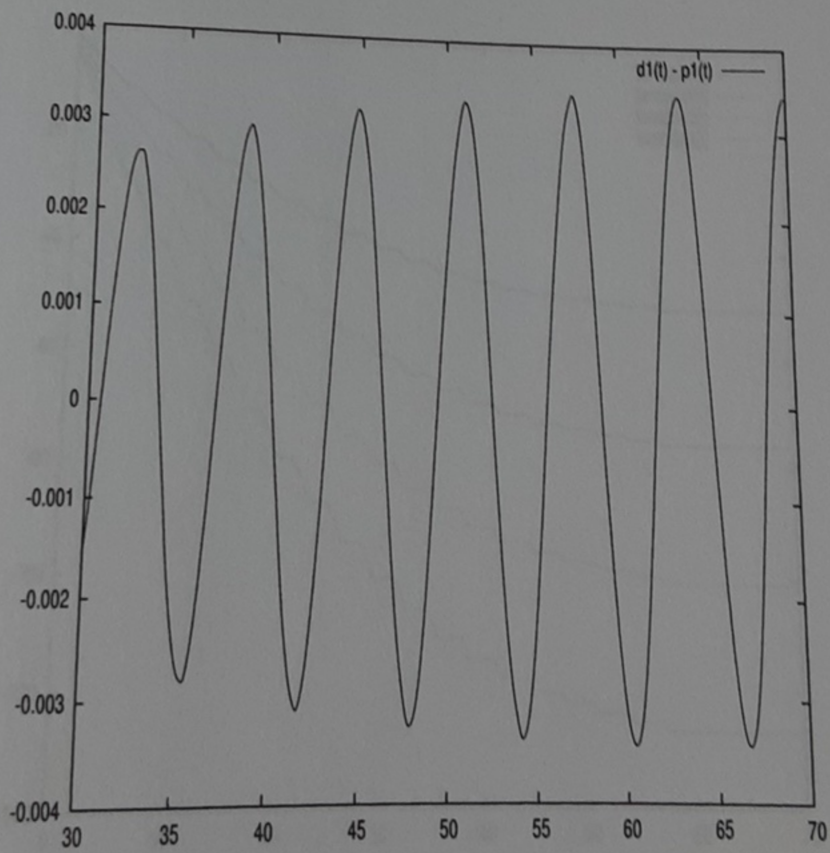

Figure 2.6.13: The difference between the actual disturbance and estimated disturbance, $d_1(t) - p_1(t)$ at some later time interval.

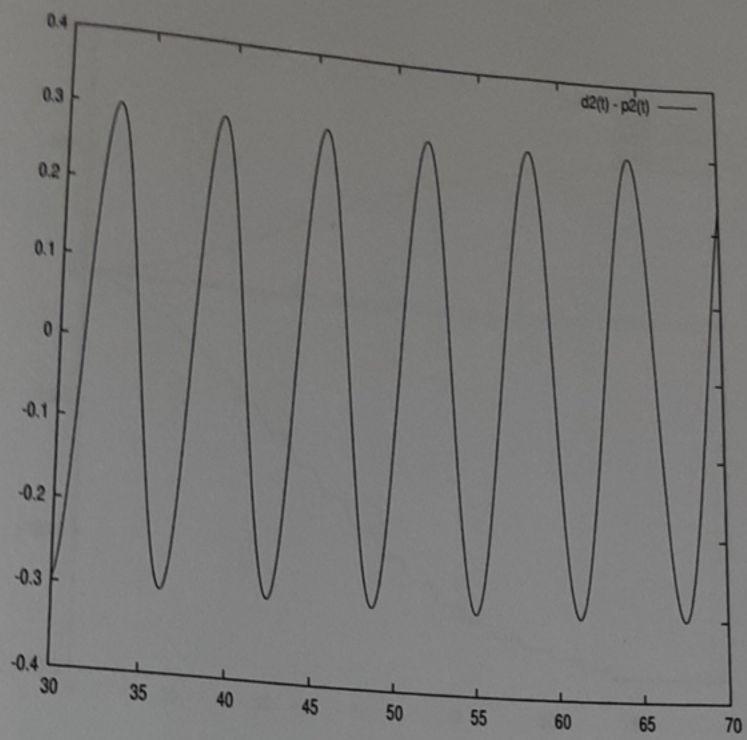

Figure 2.6.14: The difference between actual disturbance and estimated disturbance, $d_2(t) - p_2(t)$ at some later time period.

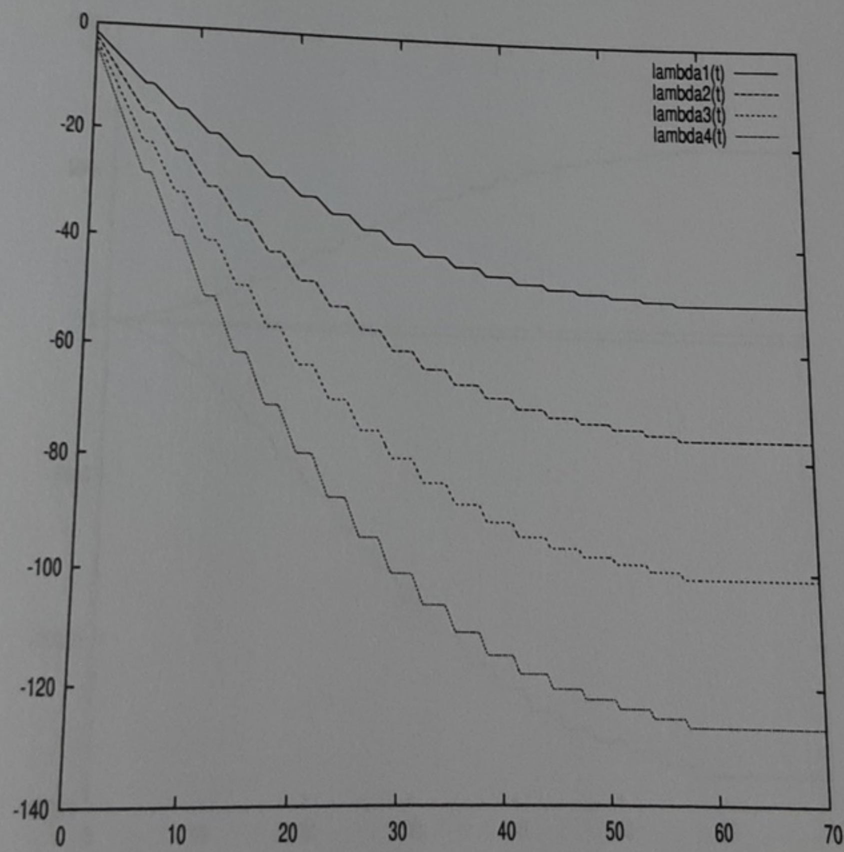

Figure 2.6.15: Histories of the eigenvalues of the error system $\lambda_1(t)$ and $\lambda_2(t)$, $\lambda_3(t)$ and $\lambda_4(t)$.

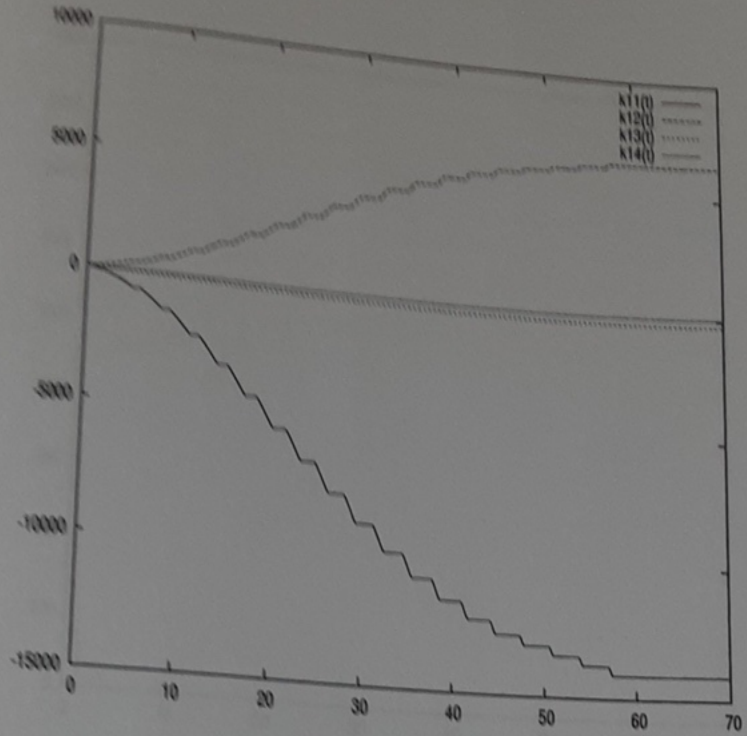

Figure 2.6.16: Histories of the feedback gains of the observer-like system $k_{11}(t)$, $k_{12}(t)$, $k_{13}(t)$ and $k_{14}(t)$.

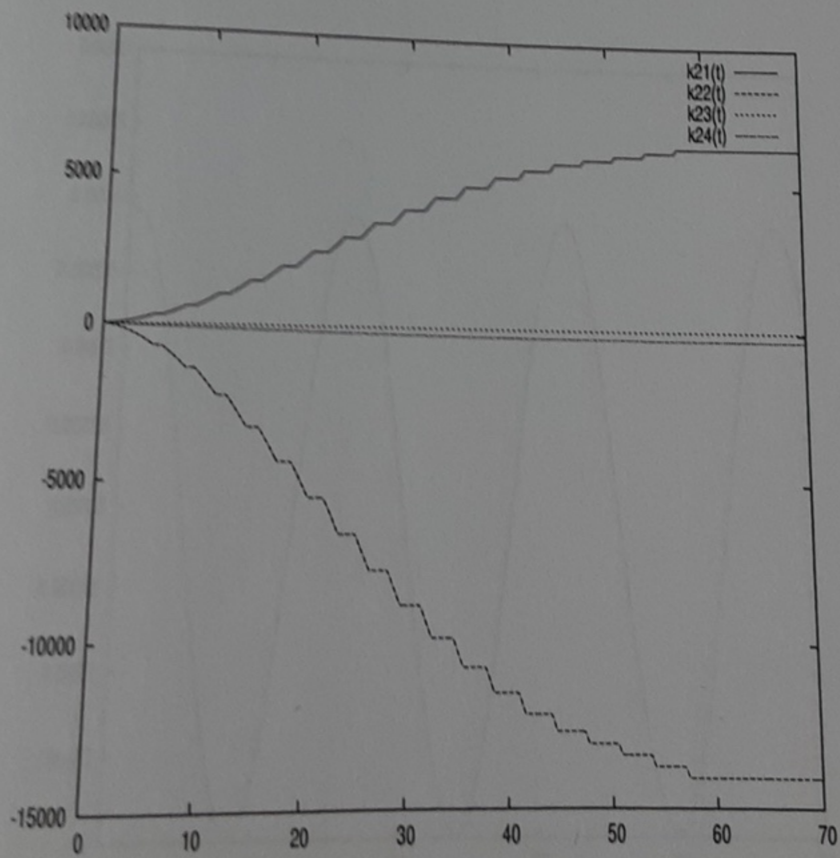

Figure 2.6.17: Histories of the feedback gains of the observer-like system $k_{21}(t)$, $k_{22}(t)$, $k_{23}(t)$ and $k_{24}(t)$.

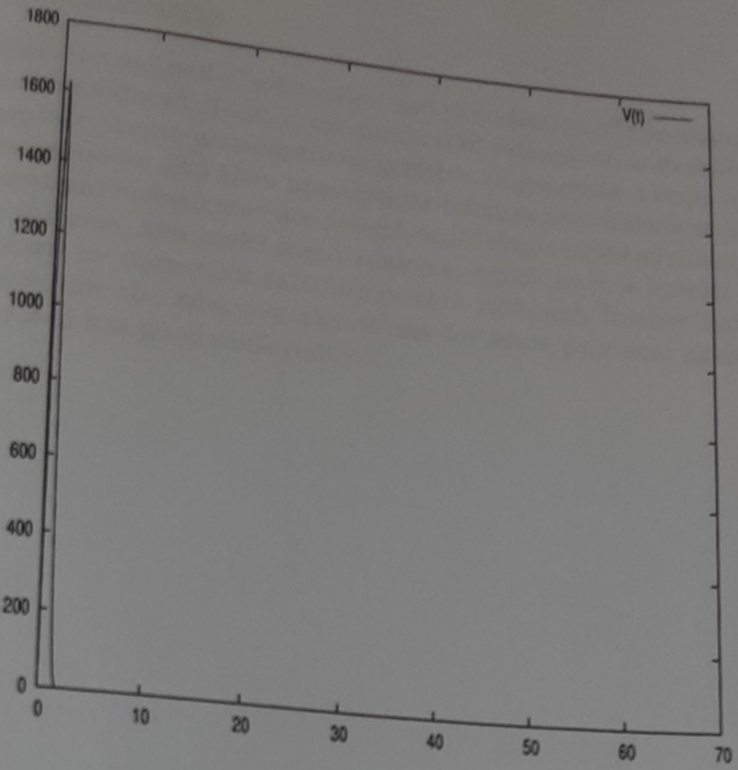

Figure 2.6.18: History of the Lyapunov-like function $V(t)$.

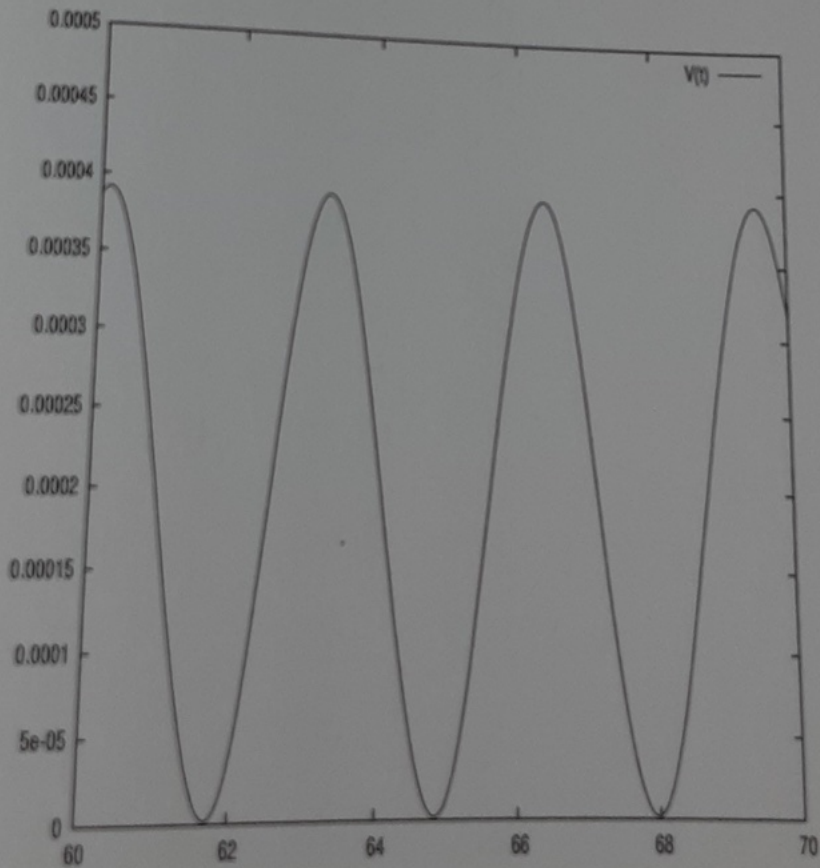

Figure 2.6.19: History of the Lyapunov-like function $V(t)$ at a later time.

2.7 Conclusions

In this chapter, the method of estimation and cancellation of uncertainty and/or disturbance is introduced. Firstly, the method of estimation is described. Next, for each class of systems, an adaptive algorithm to generate a class of adaptive controllers is proposed and then appropriate lemmas and theorems are investigated. The systems investigated are second-order single-input systems, n^{th} order single-input systems, and multi-input systems, which have a special structure. It is also shown by numerical calculation that although further improvement will be required for the adaptive algorithms for some practical situations, the method proposed has good performance.

Chapter 3

Applications of disturbance estimation and cancellation

3.1 Introduction

In this chapter, applications of estimation and cancellation of disturbance are investigated based on the methods and theorems developed in the previous chapter. The method presented in the previous chapter will be useful to hold robustness of the controlled system due to its simpleness of the method. However, there are some topics which should be investigated to make the method rigorous. For example, some important questions relating to the method are as follows. 1. Is the controlled system really robust? Since the disturbance and uncertainty can be cancelled out, it is expected that the controlled system is robust. However, it cannot be proved. 2. Is it possible to estimate the uncertainty and/or disturbance by only output measurement? The method requires all the states of the system to estimate the uncertainty and/or disturbance. Since, in practice, these assumptions cannot sometimes be satisfied, it is very useful to show that these conditions are redundant. Thus, these two problems, namely the robust property of the controlled system and output control, should be investigated. In addition, treatment of input uncertainty and unmodelled dynamics, residual uncertainty/disturbance, and the method of extraction of parameter variation from the estimated disturbance are presented.

3.2 Robustness of controlled system

In this section, the adaptive algorithm, developed in this work, is modified so that the system has a robustness property (see Definition 23 for the definition of robustness of a controlled system). Using the method of estimation and cancellation of disturbance presented in the previous chapter, any effect due to matched uncertainty and disturbance is eliminated from the system to within a specified accuracy in the transformed coordinates. Since most of the effect of uncertainty and disturbance can be cancelled out, it is expected that the controlled system is robust. However, in the original coordinates, nothing cannot be said about the robustness of the controlled system due to the fact that the accuracy specified is only in terms of the transformed coordinates. To overcome this problem, the adaptive algorithm is modified so that robustness of the controlled system can be guaranteed. The outline of this section is as follows. Firstly, a modified adaptive algorithm is introduced. Next, a definition of robustness of the system is given. Finally, lemmas and theorem which guarantee robustness of the controlled system are provided.

3.2.1 An adaptive algorithm which guarantees robustness

In this subsection, a modified adaptive algorithm which guarantees robustness of the controlled system is given. Basically, this algorithm has the same form as the algorithms given in the Chapter 2. For the modified algorithm, $||e(t)||$ is used to define $V(t)$, instead of $||\bar{e}(t)||$. This small modification plays an important role in guaranteeing robustness of the controlled system.

Algorithm 4 *i) One of the eigenvalues of the multi-input error-system (2.3.2), which is denoted by $\lambda_1(t)$, is determined as follows. Suppose δ and ϵ_e are specified constants, which are determined by control designer. Define $V(t) := ||e(t)||^2$. At $t = t_0$, $\lambda_1(t) := -\lambda_0$ and $\dot{\lambda}_1(t) := -\lambda_{d0}$, where $\lambda_0 \in \mathbb{R}^+$ and $\lambda_{d0} := 0$ or δ are chosen by the control designer. The structure of $\dot{\lambda}_1(t)$ is determined as follows:*

1. *let $\tau := t$ $(t \geq t_0)$;*
2. *evaluate $\bar{e}(\tau)$ and, hence, $V(\tau)$ is also obtained;*
3. (a) *if $(V(\tau) \leq \epsilon_e^2$ and $\dot{\lambda}_1(\tau) = -\delta)$ then $\tau_1 = \tau$ and $\dot{\lambda}_1(\cdot)$ has the following structure: $\dot{\lambda}_1(s) = f(s, \tau_1)$ for $s \geq \tau_1$;*
   (b) *or if $(V(\tau) > \epsilon_e^2$ and $\dot{\lambda}_1(\tau) = 0)$ then $\tau_1 = \tau$ and $\dot{\lambda}_1(\cdot)$ has the structure: $\dot{\lambda}_1(s) = g(s, \tau_1)$ for $s \geq \tau_1$;*
   (c) *otherwise, the structure of $\dot{\lambda}_1(\cdot)$ is not changed;*
4. *$t = t + \Delta t$ where Δt is a prescribed positive constant;*
5. *evaluate $\dot{\lambda}_1(t)$ using the given structure of $\dot{\lambda}_1(s)$.*

The functions $t \mapsto f(t, \tau)$ and $t \mapsto g(t, \tau)$ are defined by

$$f(t,\tau) := \begin{cases} \frac{1}{2}\delta \sin(\omega(t-\tau) + \frac{3}{2}\pi) - \frac{1}{2}\delta, & \tau \leq t \leq \frac{\pi}{\omega} + \tau \\ 0, & \frac{\pi}{\omega} + \tau < t \end{cases}$$

$$g(t,\tau) := \begin{cases} \frac{1}{2}\delta \sin(\omega(t-\tau) + \frac{1}{2}\pi) - \frac{1}{2}\delta, & \tau \leq t \leq \frac{\pi}{\omega} + \tau \\ -\delta, & \frac{\pi}{\omega} + \tau < t \end{cases}$$

where ω is specified constant.

ii) The remaining eigenvalues of the multi-input error-system, which are denoted by $\lambda_2(t) \cdots \lambda_n(t)$, are determined as follows:

$$\lambda_i(t) = \kappa_i \lambda_1(t),$$

where κ_i ($i = 2\cdots n$) are prescribed positive constants determined by control designer and $\kappa_i \neq 1$ for all i with $\kappa_i \neq \kappa_j$ for $i \neq j$.

iii) In addition to the above criteria, at the initial time $t = t_0$, the eigenvalues of the multi-input error system are assigned so that $\lambda_i(t) < -1$ for $i \geq 1$ and $t \geq t_0$.

3.2.2 Robust property of the controlled system for the stabilization problem

In this subsection, a theorem which guarantees robustness of the controlled system is presented. In this work it is shown that if the norm of the state of the error system (2.3.2) is less than a known constant, the norm of the state of the system (2.2.5) is also less than the same constant.

Definition 23 *System (2.2.5) is robust with respect to parametric uncertainty and external disturbance if the state of the system is driven into a certain region and the 'size' of that region can be made as small as desired by tuning a control design parameter.*

Ideally, it is desired that any effect of disturbance be cancelled out and, hence, the system is asymptotically stable. However, by nature of this method, although nearly exact knowledge of the disturbance can be estimated, it is impossible to make the system asymptotically stable as a consequence of the characteristics of the dynamics of the error system. Therefore, the condition that the states are driven to their equilibriums is relaxed in the sense of Definition 23. Note that this property of the system is sometimes called *practical stability* (see [37]). In practice, however, this method is still useful since most of the effect of uncertainty/disturbance can be cancelled out. A detailed analysis of this property is given by the following lemmas and theorem, namely Lemmas 21-24 and Theorem 6.

Lemma 21 *For both single-input and multi-input systems, there exists a positive constant κ such that the inequality*

$$\frac{1}{\kappa|\lambda_1^{n-1}(t)|} \leq \frac{1}{||T(t)||_1}, \quad \forall t \geq t_0,$$

holds, where n is the dimension of the system, $\lambda_1(t)$ is determined as described in Algorithm 4 and $T(t)$ is defined in (2.5.4) or (2.6.5).

Proof The single-input and multi-input cases are considered separately.

i) Single-input case.

Recall that $T(t)$, for n^{th} order single-input system, is given as follows (see (2.5.4)):

$$T(t) = \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1(t) & \cdots & \lambda_n(t) \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1}(t) & \cdots & \lambda_n^{n-1}(t) \end{bmatrix}.$$

Hence, the 1-norm of $T(t)$ is

$$||T(t)||_1 = \sum_i^n 1 + \cdots + \sum_i^n |\lambda_i^{n-1}(t)|.$$

As a result of Algorithm 4, $\lambda_i(t) = \kappa_i\lambda_1(t)$ and $|\lambda_i(t)| > 1$ for all $i \geq 1$ and $\kappa_1 = 1$. Thus, for all $t \geq t_0$,

$$\begin{aligned} ||T(t)||_1 &= \sum_{i=1}^{n} 1 + \cdots + \sum_{i=1}^{n} \kappa_i |\lambda_1^{n-1}(t)| \\ &\leq |\lambda_1^{n-1}(t)| \sum_{i=1}^{n} \kappa_i + \cdots + |\lambda_1^{n-1}(t)| \sum_{i=1}^{n} \kappa_i \\ &= |\lambda_1^{n-1}(t)|\kappa, \end{aligned}$$

where $\kappa = n \sum_i^n \kappa_i$.

ii) Multi-input case.

Recall that, for the multi-input system, $T(t)$ is represented as follows (see (2.6.5)):

$$T(t) = \begin{bmatrix} T_{11} & \cdots & T_{1n} \\ T_{11}\Lambda_1(t) & \cdots & T_{1n}\Lambda_n(t) \\ \vdots & \ddots & \vdots \\ T_{11}\Lambda_1^{n-1}(t) & \cdots & T_{1n}\Lambda_n^{n-1}(t) \end{bmatrix},$$

where T_{1i} are constant matrices, $\Lambda(t) = diag(\Lambda_1(t), \Lambda_2(t), \cdots, \Lambda_n(t))$, and $\Lambda_i(t)$ are submatrices of $\Lambda(t)$ in appropriate dimensions. Since, by Algorithm 4, $|\lambda_i(t)| > 1$, then the 1-norm of $T(t)$ is given by

$$||T(t)||_1 = \sum_{i}^{n} ||T_{1i}||_1 + \cdots + \sum_{i}^{n} ||T_{1i}\Lambda_i^{n-1}(t)||_1.$$

Recall that $\Lambda(t)$ is expressed as follows:

$$\begin{aligned} \Lambda(t) &= diag(\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t)) \\ &= \lambda_1(t) diag(\kappa_1, \kappa_2, \cdots, \kappa_n) \\ &= \lambda_1(t) K_\kappa \end{aligned}$$

where $\kappa_1 = 1$ and $K_\kappa = diag(\kappa_1, \kappa_2, \cdots, \kappa_n)$. Thus, $T_{1i}\Lambda_i^{n-1}(t)$ is given by,

$$T_{1i}\Lambda_i^{n-1}(t) = \lambda_1^{n-1}(t) T_{1i} K_{\kappa i} \tag{3.2.1}$$

Hence, there exists a positive constant κ such that

$$||T(t)||_1 \leq \kappa |\lambda_1^{n-1}(t)| \tag{3.2.2}$$

Hence, for both the single-input and multi-input cases, the upper bound on $||T(t)||_1$ is proportional to $|\lambda_1^{n-1}(t)|$. Hence, for both the single-input and multi-input systems,

$$\frac{1}{||T(t)||_1} \geq \frac{1}{\kappa |\lambda_1^{n-1}(t)|}, \quad \forall t \geq t_0, \tag{3.2.3}$$

where κ is some positive constant. ■

Lemma 22 *There exist positive constants k_1 and k_2 such that following relation holds*

$$\epsilon(t) \leq \frac{1}{k_1 |\lambda_1^n(t)| - k_2 |\lambda_1^{n-2}(t)|},$$

where $\epsilon(t)$ is defined in (2.5.17) for the single-input system and (2.6.15) for the multi-input system and $\lambda_1(t)$ is specified in Algorithm 4.

Proof The single-input and multi-input cases are considered separately.

i) Single-input case.

By (2.5.17),

$$\epsilon(t) = -\frac{\alpha||T^{-1}(t)B||}{\sigma_{\max}(\Lambda(t)) + ||T^{-1}(t)\dot{T}(t)||_1}.$$

Now,

$$\sigma_{\max}(\Lambda(t)) = \kappa_i\lambda_1(t),$$

for some $i \in \{1, 2, \cdots, n\}$ and $\kappa_1 = 1$. Also, using (2.5.9) and (2.5.10)

$$\begin{aligned}
||T^{-1}(t)B|| &\leq \frac{1}{\alpha_n}\frac{1}{\sqrt{\lambda_1^{2(n-1)}(t) + \cdots + \lambda_n^{2(n-1)}(t)}} \\
&= \frac{1}{\alpha_n}\frac{1}{\sqrt{\lambda_1^{2(n-1)}(t) + \cdots + \kappa_n^{2(n-1)}\lambda_1^{2(n-1)}(t)}} \\
&= \frac{1}{\bar{K}}\frac{1}{|\lambda_1^{n-1}(t)|},
\end{aligned} \tag{3.2.4}$$

where $\bar{K} = \alpha_n\sqrt{1 + \kappa_2^{2(n-1)} + \cdots + \kappa_n^{2(n-1)}}$ and α_n is a positive constant. In addition, by (2.5.14),

$$||T^{-1}(t)\dot{T}(t)||_1 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(k-1)\xi_{ik}(t)\dot{\lambda}_j(t)\frac{\lambda_j^{k-2}(t)}{\sqrt{\lambda_1^{2(k-1)} + \cdots + \lambda_n^{2(k-1)}(t)}}.$$

Since $|\xi_{ik}(t)|$ and $|\dot{\lambda}_j(t)|$ are bounded by constants, there exist positive constants γ_i such that

$$\begin{aligned}
||T^{-1}(t)\dot{T}(t)||_1 &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_i\frac{\kappa_j^{k-2}\lambda_1^{k-2}(t)}{\lambda_1^{k-1}(t)\sqrt{1 + \kappa_1^{2(k-1)} + \cdots + \kappa_n^{2(k-1)}}} \\
&= \gamma\frac{1}{|\lambda_1(t)|},
\end{aligned}$$

where γ is some positive constant. Hence, there exist positive constants k_1 and k_2 such that

$$\epsilon(t) \leq \frac{1}{k_1|\lambda_1^n(t)| - k_2|\lambda_1^{n-2}(t)|}.$$

where k_1 and k_2 satisfy the relation $k_2 < k_1|\lambda_1(t)|^2$.

ii) Multi-input case.

By (2.6.15),

$$\epsilon(t) = -\frac{\alpha||T^{-1}(t)B||_1}{\sigma_{\max}(\Lambda(t)) + ||T^{-1}(t)\dot{T}(t)||_1}.$$

As for the single-input case,

$$\sigma_{\max}(\Lambda(t)) = \kappa_i\lambda_1(t),$$

for some $i \in \{1, 2, \cdots, n\}$ and $\kappa_1 = 1$. Also by (2.6.13),

$$||T^{-1}(t)B||_1 = \eta \frac{1}{|\lambda_1(t)|^{n-1}},$$

where η is some positive constant. In addition, by (2.6.14),

$$||T^{-1}(t)\dot{T}(t)||_1 \leq \sum_{i=1}^{n}\sum_{j=1}^{n} |\frac{1}{\lambda_1(t)} C_{i2}(t)\tilde{T}_{2j}(t)| + \cdots + |(n-1)\frac{1}{\lambda_1(t)} C_{in}(t)\tilde{T}_{nj}(t)|.$$

Since $C_{ij}(t)$ and $\tilde{T}_{ij}(t)$ are bounded, there exists positive constant $\bar{\eta}$ such that

$$||T^{-1}(t)\dot{T}(t)||_1 \leq \bar{\eta}\frac{1}{|\lambda_1(t)|}$$

Hence, there exist positive constants k_1 and k_2 such that

$$\epsilon(t) \leq \frac{1}{k_1|\lambda_1^n(t)| - k_2|\lambda_1^{n-2}(t)|}.$$

where k_1 and k_2 satisfy the relation $k_2 < k_1|\lambda_1(t)|^2$.

Therefore, for both the single-input and multi-input cases, the upper bound of $\epsilon(t)$ is represented as follows:

$$\epsilon(t) \leq \frac{1}{k_1|\lambda_1^n(t)| - k_2|\lambda_1^{n-2}(t)|}. \tag{3.2.5}$$

■

In the following lemma, analogous to Lemma 8, it is shown that $\epsilon(t)$ converges to a positive constant.

Lemma 23 *If $\epsilon(t)$ converges as $t \to \infty$, where $\lambda_i(t)$ satisfy the conditions in Algorithm 4, then there exists a positive constant ν such that $\lim_{t\to\infty} \epsilon(t) = \nu$.*

Proof The statement of this lemma is proved using a contradiction argument. Assume $\lim_{t\to\infty} \epsilon(t) = 0$. Then, by Algorithm 4, $|\lambda_1(t)|$ is a non-decreasing function and tends to infinity. Therefore, there exists $t_2 > t_1$ such that

$$|\lambda_1(t_2)| < \frac{1}{k_4^{\frac{1}{n-1}}}|\lambda_1(t_1)| \left[k_1|\lambda_1(t_1)| - k_2|\lambda_1(t_1)|^{-1}\right]^{\frac{1}{n-1}},$$

where $k_4 = \frac{k_3}{\epsilon_e}$ and k_3 is positive constant. Since $\lambda_1(t) \to -\infty$ as $t \to \infty$, as a result of this inequality, for t_2 sufficiently large, $|\lambda_1(t_2) - \lambda_1(t_1)|$ can be arbitrarily large. Hence,

$$\frac{1}{k_1|\lambda_1(t_1)|^n - k_2|\lambda_1(t_1)|^{n-2}} < \frac{1}{k_3|\lambda_1(t_2)|^{n-1}}\epsilon_e,$$

and so, using Lemma 21 and 22,

$$\epsilon(t_1) \leq \frac{1}{k_1|\lambda_1(t_1)|^n - k_2|\lambda_1(t_1)|^{n-2}} < \frac{1}{k_3|\lambda_1(t_2)|^{n-1}}\epsilon_e \leq \frac{1}{||T(t_2)||_1}\epsilon_e.$$

Since $\lambda_1(t) \to -\infty$ as $t \to -\infty$, the conditions of Lemma 7 hold and then, in view of Lemma 7, there exists $t_2 > t_3$ such that $\bar{e}(t) \in \mathbb{B}(\epsilon(t_1))$ for $t > t_3$. Therefore, for $t > t_3$ and t_2 sufficiently large,

$$||\bar{e}(t)|| \leq \epsilon(t_1) < \frac{1}{||T(t_2)||}\epsilon_2 \quad (3.2.6)$$

and

$$|\lambda_1(t_1)| < |\lambda_1(t_3)| < |\lambda_1(t_2)|, \quad t_2 > t_3 > t_1. \quad (3.2.7)$$

Since $||e(t)|| \leq ||T(t)||_1||\bar{e}(t)||$, for $t_2 > t > t_3$ and t_2 sufficiently large,

$$\begin{aligned} ||e(t)|| &\leq ||T(t)||_1\epsilon(t_1) \\ &< ||T(t)||_1\frac{1}{||T(t_2)||_1}\epsilon_e. \end{aligned}$$

Since $|\lambda_1(t_2)| > |\lambda_1(t)| > |\lambda_1(t_3)|$ and $||T(t)||_1$ is a non-decreasing function,

$$\begin{aligned} ||e(t)|| &< ||T(t)||_1\frac{1}{||T(t_2)||_1}\epsilon_e \\ &< \epsilon_e, \quad t_2 > t > t_3. \end{aligned} \quad (3.2.8)$$

Since, t_2 can be arbitrarily large, as a consequence of inequality (3.2.8) and Algorithm 4, there exists t_3' such that $||e(t)|| < \epsilon_e$, $|\lambda_1(t_2)| > |\lambda_1(t_3')|$, and $\dot{\lambda}_1(t) = 0$ hold for $t_2 > t > t_3' > t_3$. Hence, $||T(t)||_1 = ||T(t_3')||_1$ for $t_2 > t > t_3'$. $||e(t)|| < \epsilon_e$ for $t_2 > t > t_3'$ implies that $||e(t_2)|| \leq \epsilon_e$. It follows that $||T(t_2)|| = ||T(t_3')||$, which contradicts $|\lambda_1(t_2)| > |\lambda_1(t_3')|$. Therefore, the assumption $\lim_{t\to\infty}\epsilon(t) = 0$ is invalid. Therefore, there exists positive constant ν such that $\lim_{t\to\infty}\epsilon(t) = \nu$ holds. ■

Remark 30 Although the analysis of this proof and Lemma 7 assume different algorithms, the result of Lemma 7 can be still used within this proof since the conditions specified in Algorithm 4 satisfy the property required for Lemma 7.

Lemma 24 *If $\lambda_i(t)$ satisfy the conditions given in Algorithm 4, and if $\sigma_{\max}(\Lambda(t_1)) + ||T^{-1}(t_1)\dot{T}(t_1)||_1 < 0$ ($t_1 > t_0$), then for any prescribed $\epsilon_e > 0$, there exists $t_2 \geq t_1$ such that $e(t) \in \mathbb{B}(\epsilon_e)$ for $t \geq t_2$ and $\lambda_i(t)$ are finite for all $t \geq t_2$.*

Proof Using Algorithm 4 and Lemma 23 instead of using Algorithm 1 and Lemma 8, proof of Lemma 9 can be adapted to prove this lemma. ■

Theorem 6 *In the presence of parametric uncertainty and external disturbances, system (2.2.5) is robust, in the sense of Definition 23 using the disturbance estimation/cancellation method.*

Proof Define the actual and estimated disturbances as $v(t)$ and $\bar{v}(t)$, respectively[1]. For the error system (2.3.2), $\bar{v}(t) - v(t) = -(v(t) - \bar{v}(t)) = -\Delta v(t)$ is applied as input. For the system (2.2.5), the applied input is $\Delta v(t) = v(t) - \bar{v}(t)$. In other words, the input applied to the system (2.2.5) is always of opposite sign to the control input applied to the error system (2.3.2). Since both systems have the same system matrices and the amplitude of the input applied are always exactly the same, $||e(t)|| \leq \delta$ implies $||r(t)|| \leq \delta$. By Lemma 24, there exists $t^* \geq t_0$ such that $||e(t)|| \leq \epsilon_e$ holds for $t > t^*$. Therefore, there exists t^* such that $||r(t)|| \leq \epsilon_e$ is satisfied for $t > t^*$. Hence, the result follows. ■

[1] Of course, the actual disturbance is *matched* disturbance

3.2.3 Robust property of the controlled system for the tracking problem

In this subsection, it is shown that using the disturbance estimation/cancellation method, a control input can be designed for the nominal system so that the output of the system tracks a prescribed trajectory in the presence of uncertainty and/or disturbance. As seen in this section, the major advantages of the disturbance estimation/cancellation method are its simplicity; that is once the disturbance estimation/cancellation method is applied, another control input such as a tracking controller can be designed for the system without considering any effect of any uncertainty/disturbance.

The outline of this section is given as follows. Firstly, it is shown that the disturbance estimation/cancellation method is not affected by the existence of a tracking control input. Next, it is shown that using the disturbance estimation/cancellation method, if the tracking controller is designed appropriately for the nominal system, then tracking the desired trajectory can be done robustly in an appropriate sense. Finally, a numerical simulation and discussion is included.

Relationship between robust controllers and tracking controllers

In this subsection, it is shown that the disturbance estimation/cancellation method introduced in the previous chapter can be done independently of the implementation of the tracking control input.

The control input to the system (3.2.9) is defined as follows:

$$u(t) = u_r(t) + u_t(t),$$

where $u_r(t)$ is the control input for cancellation of the uncertainty/disturbance and $u_t(t)$ is the control input for tracking of the prescribed trajectory. Thus, it follows that

$$\dot{r}(t) = Ar(t) + B(u_r(t) + u_t(t) + p(t)). \tag{3.2.9}$$

Hence, the feedforward filter (2.2.8) can be expressed as

$$\begin{aligned} \dot{x}_f(t) &= Ax_f(t) + Bu(t) \\ &= Ax_f(t) + B(u_r(t) + u_t(t)), \end{aligned}$$

and so the modified reference signal $\bar{r}(t)$ is generated by

$$\dot{\bar{r}}(t) = A\bar{r}(t) + Bp(t). \tag{3.2.10}$$

Equation (3.2.10) holds regardless of whether a tracking controller is used or not. Also, observer-like system has the structure:

$$\dot{x}(t) = Ax(t) + B\bar{u}(t),$$

where $\bar{u}(t)$ is the control input to the observer-like system, which is determined adaptively. This system is, of course, not affected by the existence of the input for tracking. Therefore, regardless of the existence of the tracking controller, the design of the disturbance estimation/cancellation method is not affected.

Robustness of tracking

In this subsection, the property of robustness for the tracking problem is investigated. Firstly, the definition of robustness for the tracking is given. Next, a theorem and proof is presented for robustness of tracking. The analysis is based on the fact that the analysis of disturbance estimation/cancellation method is not affected by other control inputs and the linearity of the nominal system. Since the nominal system is linear, each control input can be treated independently. As a result of these characteristics, the analysis of this robustness property and the design of the robust tracking controller are simplified.

Definition 24 *In the presence of parametric uncertainty and external disturbance, the system with the tracking control input is said to be robust if the norm of the difference between the outputs or some of chosen states of the system and the reference signals can be made as small as desired.*

Theorem 7 *For the multi-input multi-output system (3.2.9), if the tracking controller is designed so that perfect tracking can be achieved with respect to the nominal system, and the disturbance estimation/cancellation method is applied, the system (3.2.9) with the tracking control input is robust in the sense of Definition 24.*

Proof Let

$$y(t) \;=\; Cr(t),$$

where $y(t)$ denotes the measurement outputs of the system or some of the states of the system which tracks reference signal $R(t)$. If $y(t)$ represents measurement outputs, then C is a output matrix, otherwise, it is a matrix which correlates some of the chosen states from full states $r(t)$. Note that since output tracking and state tracking are treated in unified framework, this notation is used. In the s-domain, the input-output relation for system (3.2.9) is given by

$$Y(s) = G(s)(U_t(s) + U_r(s) + P(s)), \tag{3.2.11}$$

where $G(s) = C(sI - A)^{-1}B$, $Y(s) = \mathcal{L}\{y(t)\}$, $U_t(s) = \mathcal{L}\{u_t(t)\}$, $U_r(s) = \mathcal{L}\{u_r(t)\}$, $P(s) = \mathcal{L}\{p(t)\}$, and $\mathcal{L}\{\cdot\}$ denotes taking the Laplace transform. Since the system is linear, this equation can be expressed as follows:

$$Y(s) = G(s)U_t(s) + G(s)(U_r(s) + P(s)).$$

If perfect tracking can be achieved, in the absence of disturbance, then

$$\tilde{R}(s) := G(s)U_t(s),$$

where $\tilde{R}(s) = \mathcal{L}\{R(t)\}$. Hence, taking the inverse Laplace transform of (3.2.11),

$$y(t) = R(t) + z(t),$$

where $z(t) := \mathcal{L}^{-1}\left\{C(sI - A)^{-1}B(U_r(s) + P(s))\right\}$ denotes the outputs or some of chosen states of the system (3.2.9) when the tracking controller is absent, then $z(t) = Cr_1(t)$ where $r_1(t)$ is the state of system (3.2.9) with $u_t := 0$. For t

sufficiently large and given any positive constant δ, it follows, from Theorem 6, that

$$||r_1(t)|| \leq \delta.$$

Hence, for t sufficiently large,

$$||z(t)|| \leq ||C||\delta.$$

Therefore,

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} \leq \begin{bmatrix} R_1(t) \\ \vdots \\ R_m(t) \end{bmatrix} + \begin{bmatrix} ||C||\delta \\ \vdots \\ ||C||\delta \end{bmatrix} \tag{3.2.12}$$

$$\begin{bmatrix} y_1(t) - R_1(t) \\ \vdots \\ y_m(t) - R_m(t) \end{bmatrix} \leq \begin{bmatrix} ||C||\delta \\ \vdots \\ ||C||\delta \end{bmatrix},$$

where $y_i(t)$ and $R_i(t)$ are i^{th} element of $y(t)$ and $R(t)$ respectively. It follows, for t sufficiently large, that

$$||y_i(t) - R_i(t)|| \leq ||C||\delta, \quad i = 1, \cdots, m.$$

In view of adaptive algorithm, Algorithm 4, positive constant δ can be made as small as desired and, hence, $y_i(t)$ can be made as close as desired to $R_i(t)$, for t sufficiently large. ■

3.2.4 Simulation example

In this subsection, the performance of the tracking problem, with estimation and cancellation of the disturbance, is examined by a numerical simulation. It will be shown that the method of estimation and cancellation of disturbance is simple, but effective.

Configuration

The system, to be examined, is a linear system with two inputs, which is modelled as

$$\dot{r}(t) = Ar(t) + B(u(t) + d(t)),$$

where $d(t) \in \mathbb{R}^2$ is the uncertainty/disturbance, $u(t) \in \mathbb{R}^2$ is the vector of the control inputs, and the system matrices are given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & -0.1 & -5 & -0.05 \\ -0.2 & -20 & -0.1 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

An initial condition for the system is $r(t_0) = [0\ 0\ 0\ 0]^t$. Uncertainties, affecting the system, are given as follows:

$$d_1(t) = 0.4 \sin t + 3.0r_1(t) + 0.03r_2(t) + 1.0r_3(t) + 0.01r_4(t),$$
$$d_2(t) = 0.2 \sin t + 0.01r_1(t) + 3.0r_2(t) + 0.01r_3(t) + 3.0r_4(t).$$

The problem considered is the well-known state tracking problem; that is $r_1(t)$ tracks a prescribed reference signal $R_1(t)$ and $r_2(t)$ tracks the reference signal $R_2(t)$. For simplicity, tracking is achieved by decoupling each sub-system using state feedback:

$$u_1(t) = 0.1r_2(t) + 0.05r_4(t),$$
$$u_2(t) = 0.2r_1(t) + 0.1r_3(t).$$

As a result of this state feedback, the system matrix for the real system now has the form:

$$\bar{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 0 & -5 & 0 \\ 0 & -20 & 0 & -9 \end{bmatrix}.$$

Next, the observer-like system and feedforward filter is designed with respect to the matrix $\bar{A}$. The following reference signals are used for tracking purposes:

$$R_1(t) = 1.0,$$
$$R_2(t) = 0.5.$$

The tracking control inputs are determined as follows. The input-output relation between $u_1(t)$ to $r_1(t)$ and $u_2(t)$ to $r_2(t)$ are given by

$$\begin{aligned} \tilde{r}_1(s) &= G_1(s)U_1(s) \\ &= \frac{1}{(s+2)(s+3)}U_1(s), \\ \tilde{r}_2(s) &= G_2(s)U_2(s) \\ &= \frac{1}{(s+4)(s+5)}U_2(s). \end{aligned}$$

where $\tilde{r}_i(s) = \mathcal{L}\{r_i(t)\}$ and $U_i(s) = \mathcal{L}\{u_i(s)\}$. Hence, the gains of each transfer functions with respect to constant signals are given by

$$|G_1(0)| = \frac{1}{6},$$
$$|G_2(0)| = \frac{1}{20}.$$

Hence, the tracking control inputs are determined as follows:

$$\begin{aligned} u_{t1}(t) &= 6 \cdot R_1(t) \\ &= 6, \\ u_{t2}(t) &= 20 \cdot R_2(t) \\ &= 10. \end{aligned}$$

The following parameter values are used for the adaptive algorithm: $\epsilon_e^2 = 2.0 \times 10^{-7}$, $\omega = 10$, $\delta = 2.0$, $\kappa_2 = 2.0$, $\kappa_3 = 3.0$, $\kappa_4 = 4.0$, $\dot{\lambda}_1(t_0) = 0$, and $\lambda_1(t_0) = -2.0$, where ϵ_e represents the accuracy of estimation, $\omega > 0$ ensures $\lambda_i(t)$ are continuous, and δ are used to determine the amplitudes of $\dot{\lambda}_1(t)$. Also, estimated disturbances are used to cancel out effect of disturbance to the system; i.e. the opposite sign of the estimated disturbances are fed back to the system. The simulation has been performed with the following configuration:

Programming language: C++;

Compiler: g++ version 3.2.2;

Algorithm to obtain the numerical solution of the ODE: Runge-Kutta (see [31] for details);

Time step for the Runge-Kutta method: 10^{-5};

Algorithm to obtain the inverse matrix: Gauss-Jordan Elimination (see [31] for details).

Simulation results

The open-loop response of the states of the system are shown in Figure 3.2.1 to 3.2.4. In Figure 3.2.1, the dashed line represents the reference signal $R_1(t)$ and the solid line represents the state $r_1(t)$. Also, in Figure 3.2.2, the dashed line represents the reference signal and the solid line represents the state $r_2(t)$. In these figures, it is clear that because of the presence of disturbance and uncertainty, tracking the reference signals are not achieved.

The closed-loop responses of the states of the system are illustrated in Figure 3.2.5 to 3.2.10. The reference signal $R_1(t)$ and the closed-loop response of $r_1(t)$ are shown in Figure 3.2.5. The dashed line represents the reference signal and the solid line represents the state $r_1(t)$. In this figure, it is observed that this state converges to the reference signal. The behaviour of the state $r_1(t)$ and the reference signal at a later time are shown in Figure 3.2.9. In this figure, it is observed that although this state is not exactly the same as the reference signal, its error is small, i.e. its order is 10^{-4} with respect to the reference signal. As it is observed in Figure 3.2.6, similar thing can be said for the state $r_2(t)$ and the reference signal $R_2(t)$. Therefore, the figures confirm that using the disturbance estimation/cancellation method, the tracking controller, designed with respect to the nominal system, can be used as a tracking controller in the presence of uncertainty.

The actual and estimated disturbances are shown in Figures 3.2.11 to 3.2.14. The actual disturbance, $d_1(t)$, and its estimate, $p_1(t)$, are shown in Figure 3.2.11. The solid line represents actual disturbance and the dashed line represents estimated disturbance. In this figure, it is observed that the estimated disturbance converges to the actual one fairly rapidly. The behaviour of these disturbances at some later time are shown in Figure 3.2.13. In this figure, it is observed that although there is error between the actual and the estimated disturbances, the error is small. In Figure 3.2.12 and 3.2.14, similar behaviour is observed for the set of the actual disturbance $d_2(t)$ and the estimated disturbance $p_2(t)$. The figures confirm that the disturbances have been estimated to the desired accuracy.

The history of the eigenvalues of the error system (2.3.2) and the gains of the observer-like system (2.2.4) are shown in Figures 3.2.15, 3.2.16, and 3.2.17. In these figures, it is observed that these values are increased or decreased until they reach certain values and then they remain at those values. The history of the Lyapunov-like function is shown in Figure 3.2.18. In this figure, it is observed that the value of this function decrease very rapidly. The history of this function, at some later time, is shown in Figure 3.2.19. In this figure, it is observed that the value of this function decreases until its value lies within a prescribed interval and, thereafter, its value remains in this interval, namely $V(t) \le \epsilon_e^2$ for $t > t^*$. Therefore, the simulation confirms the results of using the adaptive algorithm for the tracking problem.

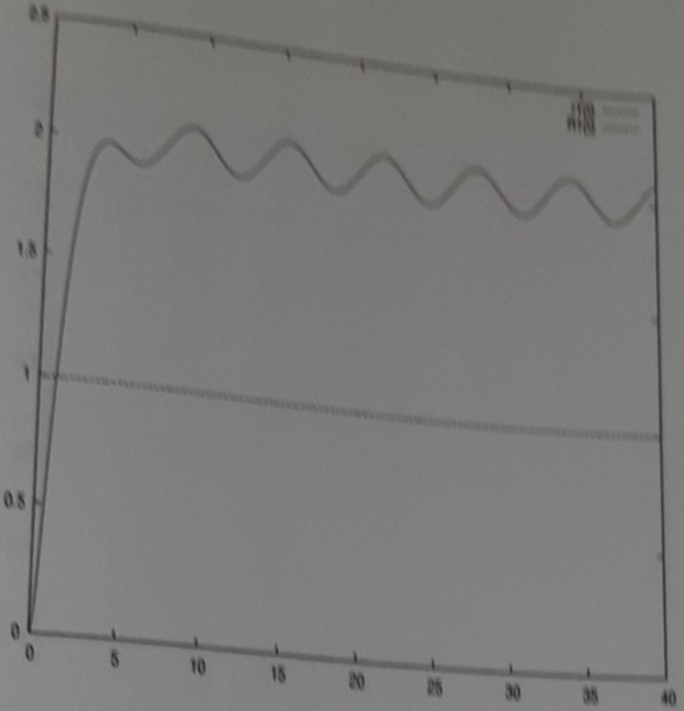

Figure 3.2.1: Open-loop response of the state $r_1(t)$ and the reference signal $R_1(t) = 1.0$.

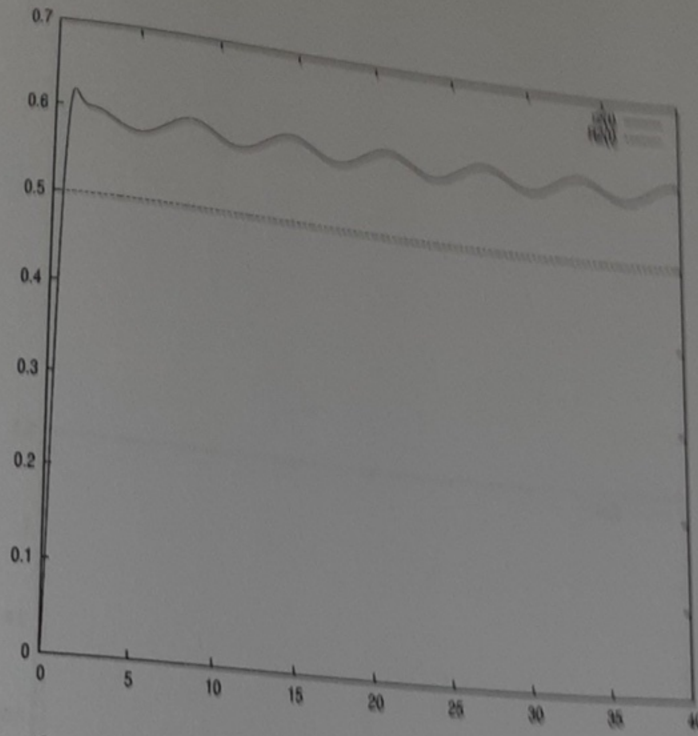

Figure 3.2.2: Open-loop response of the state $r_2(t)$ and the reference signal $R_2(t) = 0.5$.

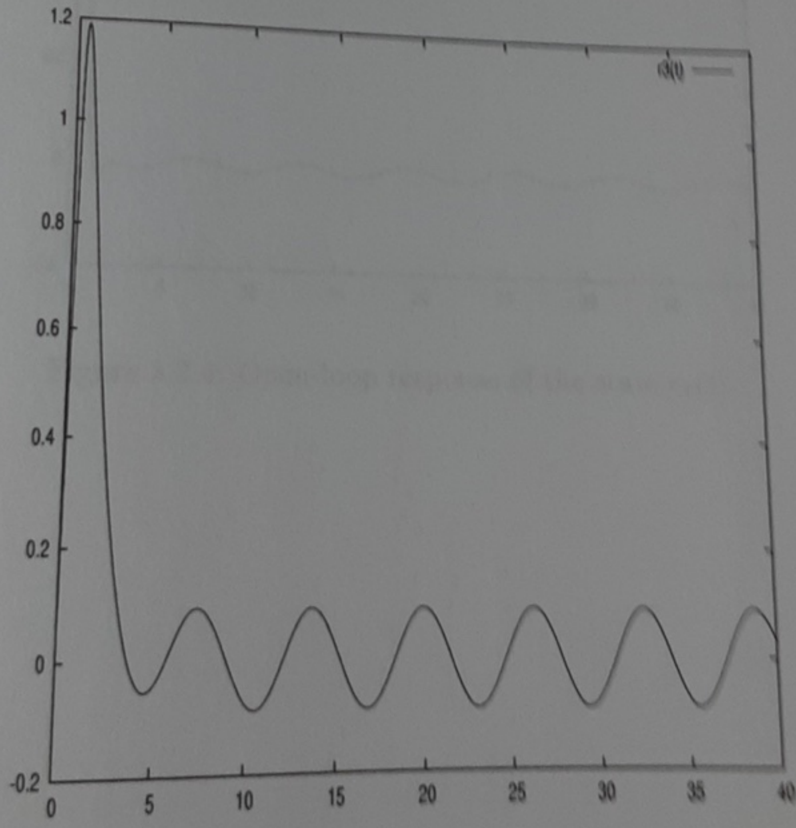

Figure 3.2.3: Open-loop response of the state $r_3(t)$.

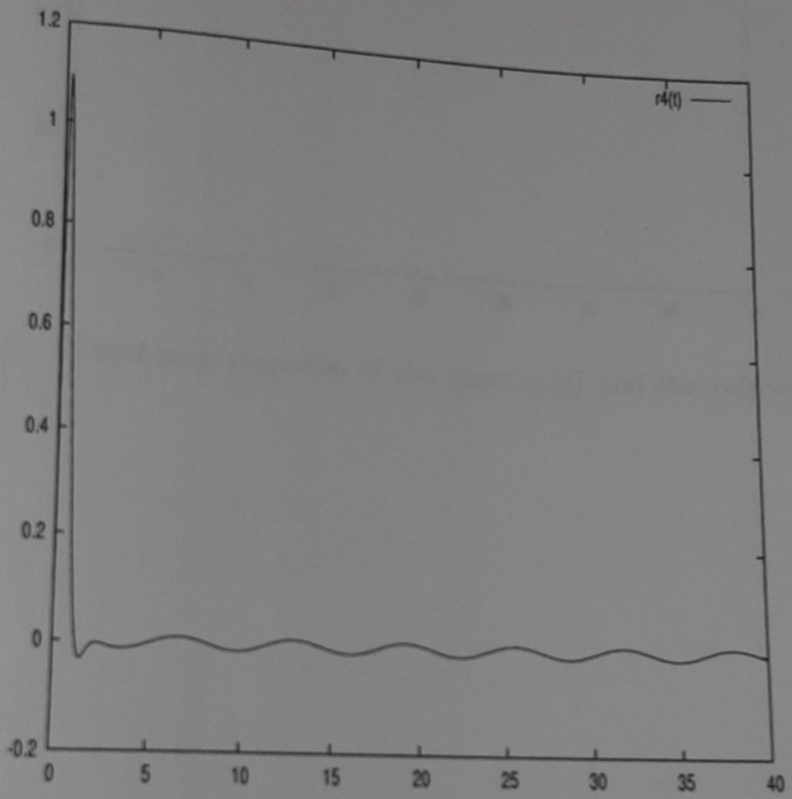

Figure 3.2.4: Open-loop response of the state $r_4(t)$.

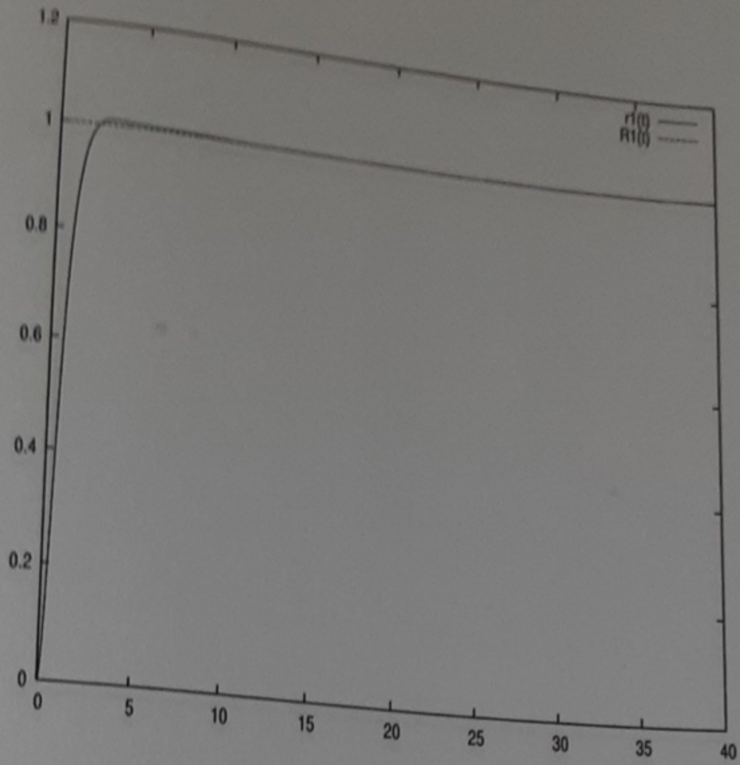

Figure 3.2.5: Closed-loop response of the state $r_1(t)$ and the reference signal $R_1(t) = 1.0$.

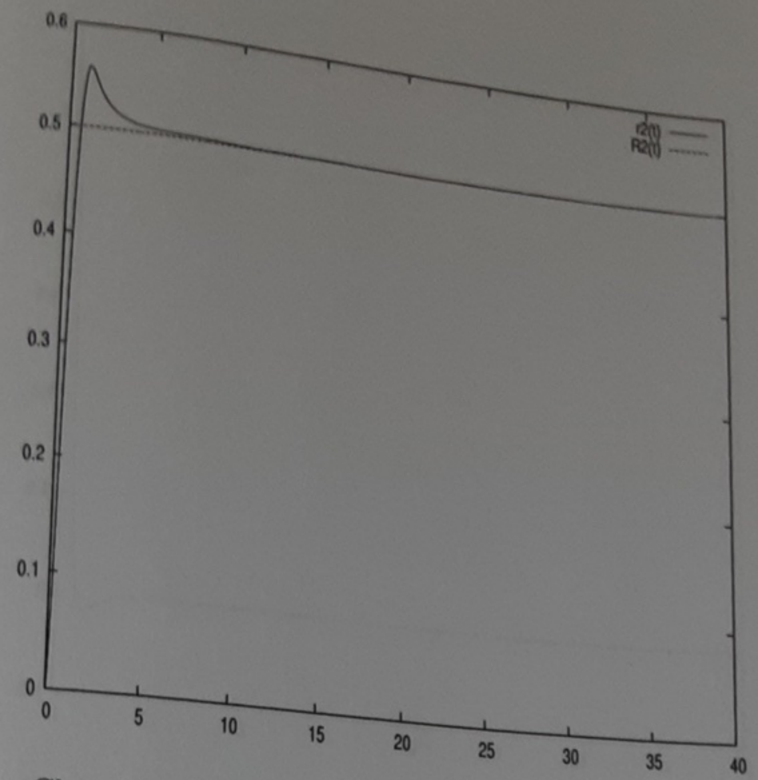

Figure 3.2.6: Closed-loop response of the state $r_2(t)$ and the reference signal $R_2(t) = 0.5$.

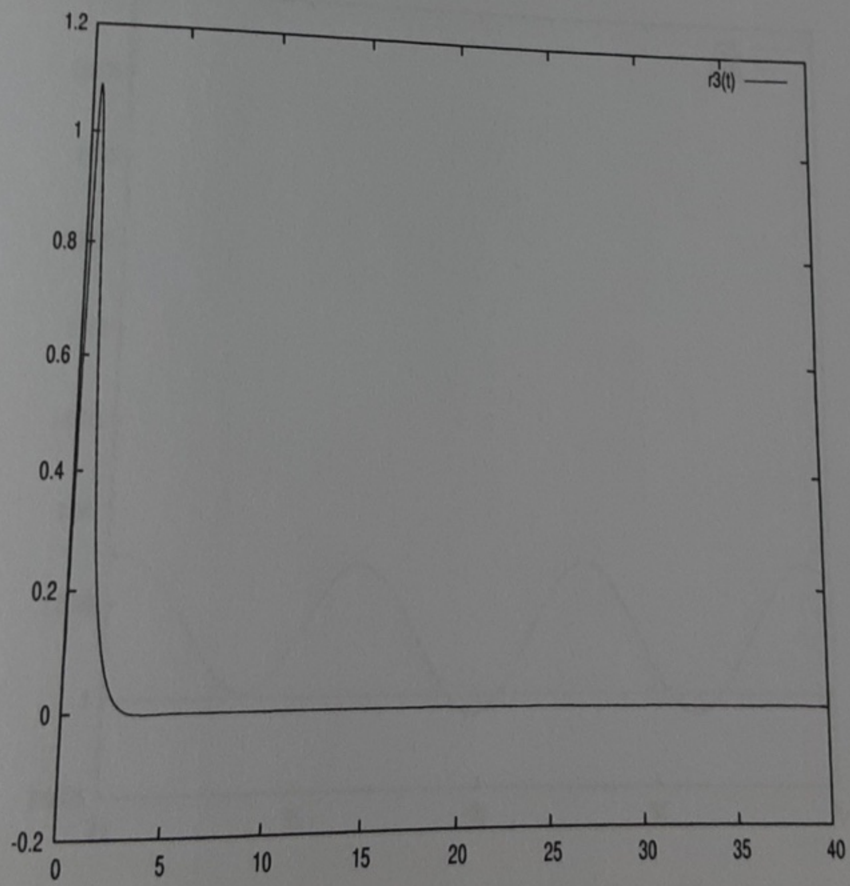

Figure 3.2.7: Closed-loop response of the state $r_3(t)$.

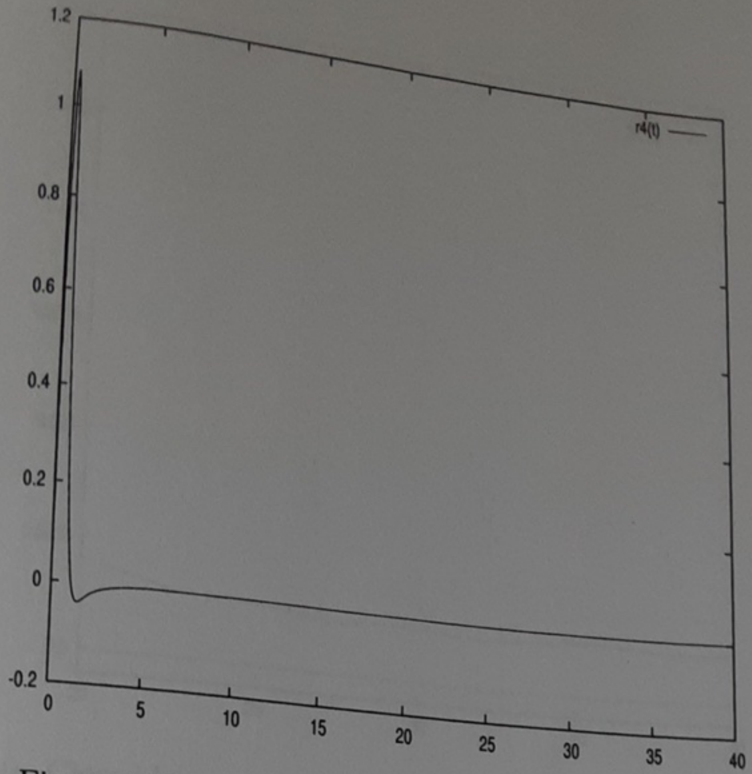

Figure 3.2.8: Closed-loop response of the state $r_4(t)$.

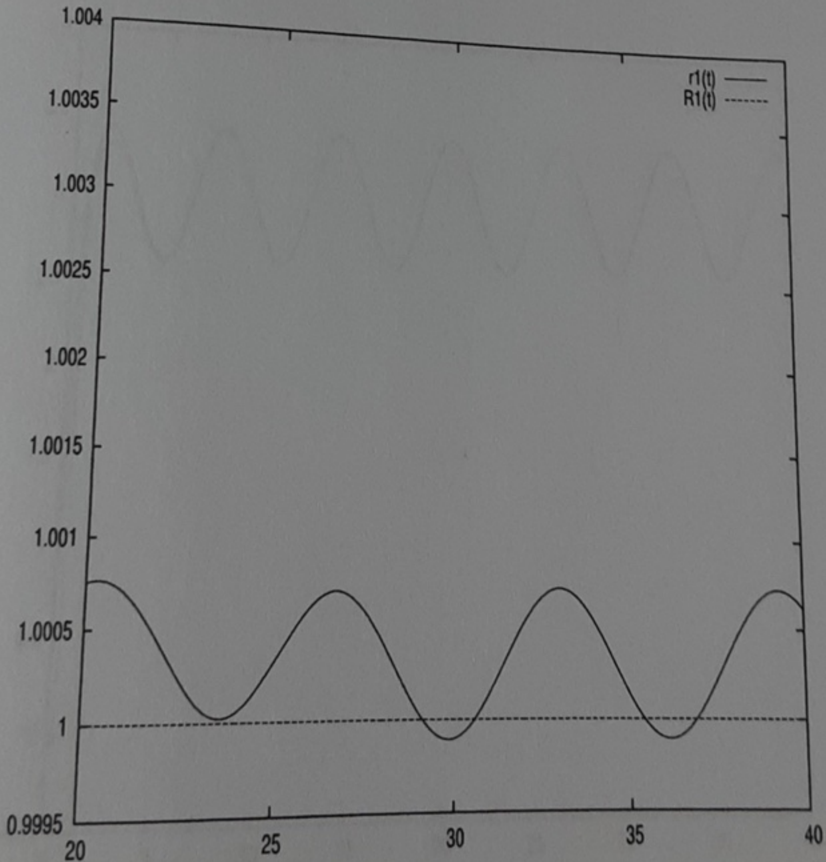

Figure 3.2.9: Closed-loop response of the state $r_1(t)$ and the reference signal $R_1(t) = 1.0$ at some later time.

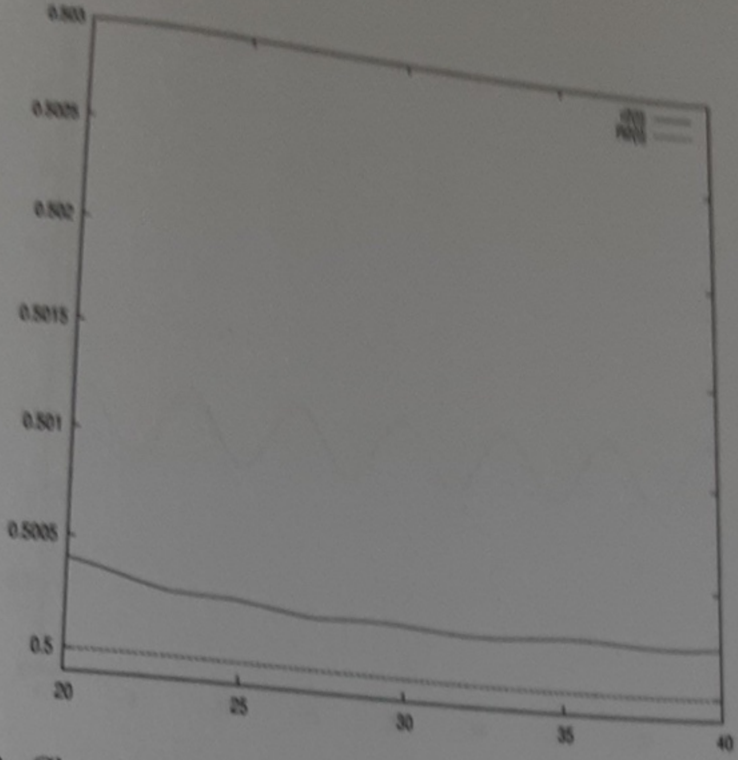

Figure 3.2.10: Closed-loop response of the state $r_2(t)$ and the reference signal $R_2(t) = 0.5$ at some later time.

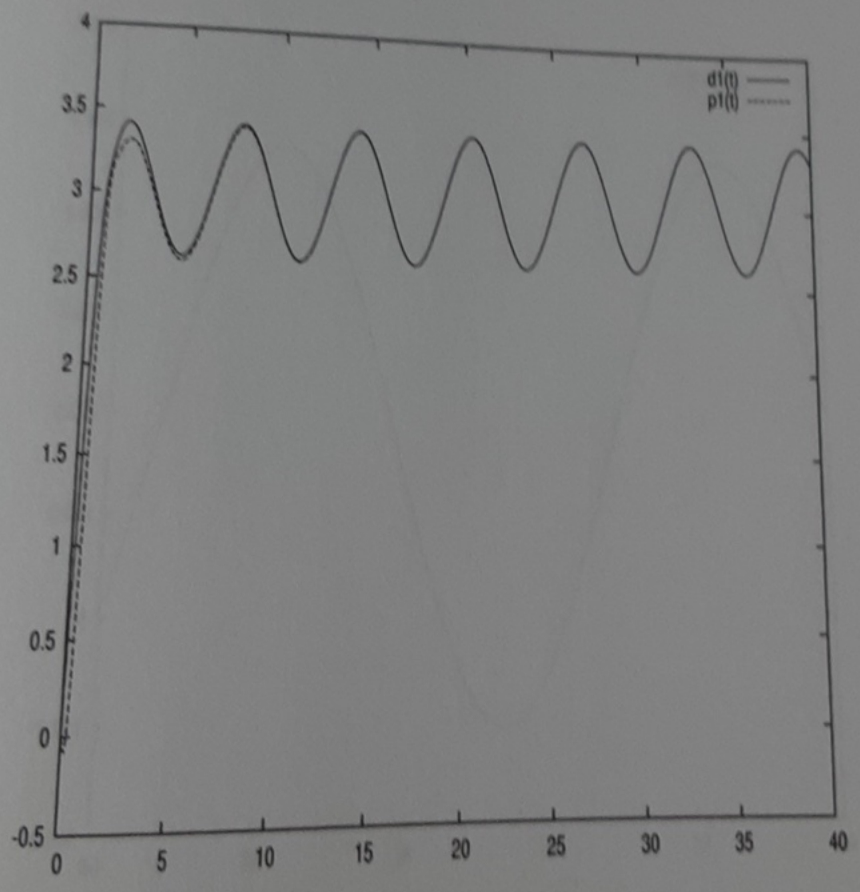

Figure 3.2.11: The actual and estimated disturbances, $d_1(t)$ and $p_1(t)$, respectively.

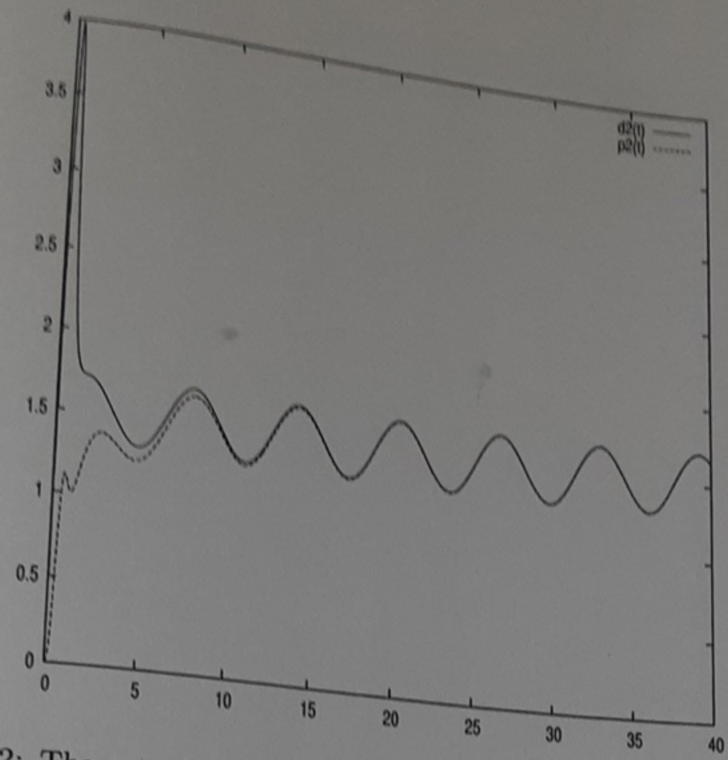

Figure 3.2.12: The actual and estimated disturbances, $d_2(t)$ and $p_2(t)$, respectively.

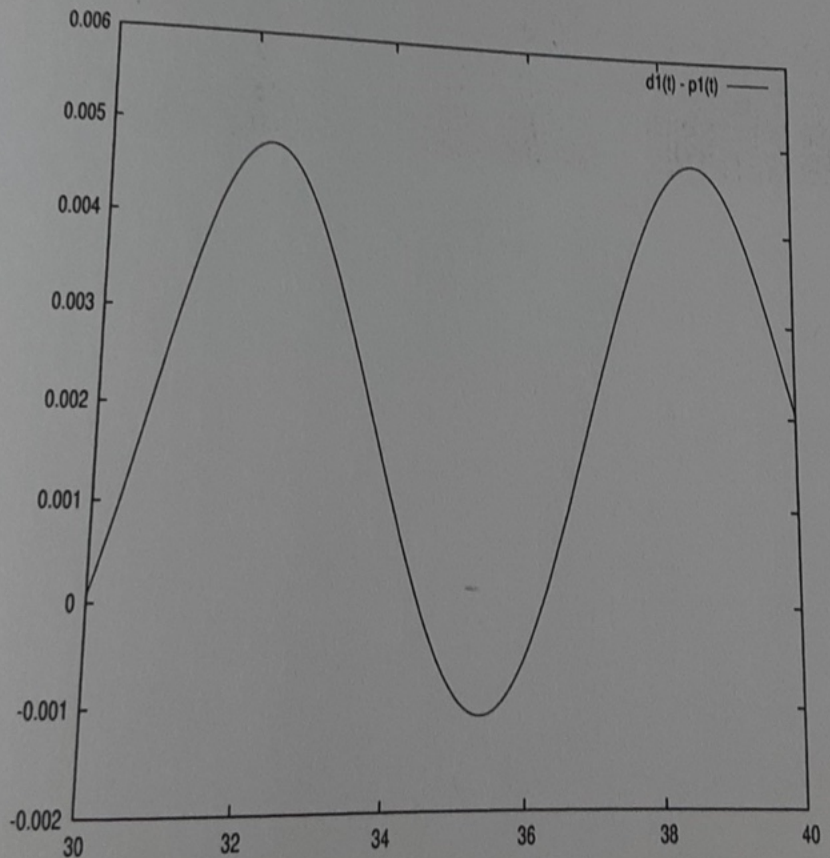

Figure 3.2.13: The actual and estimated disturbances, $d_1(t)$ and $p_1(t)$, respectively, at some later time.

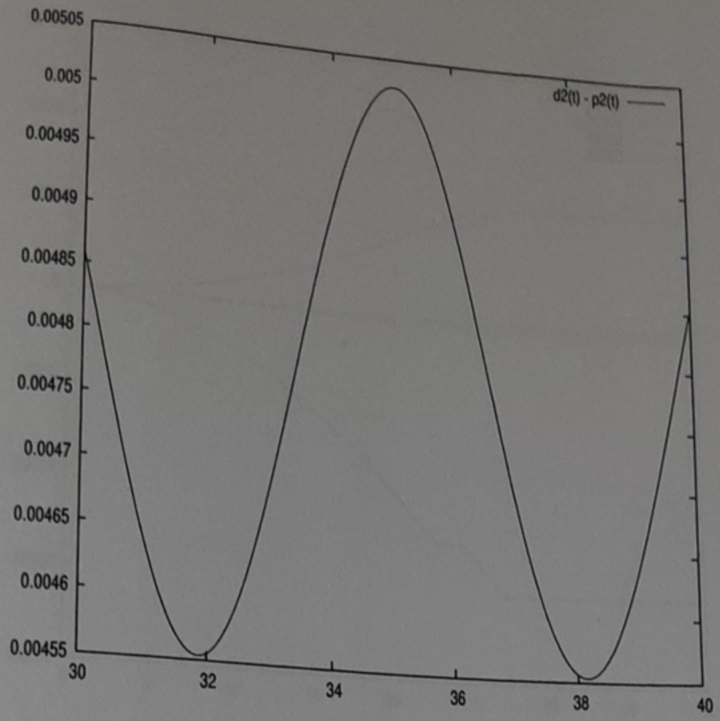

Figure 3.2.14: The actual and estimated disturbances, $d_2(t)$ and $p_2(t)$, respectively, at some later time.

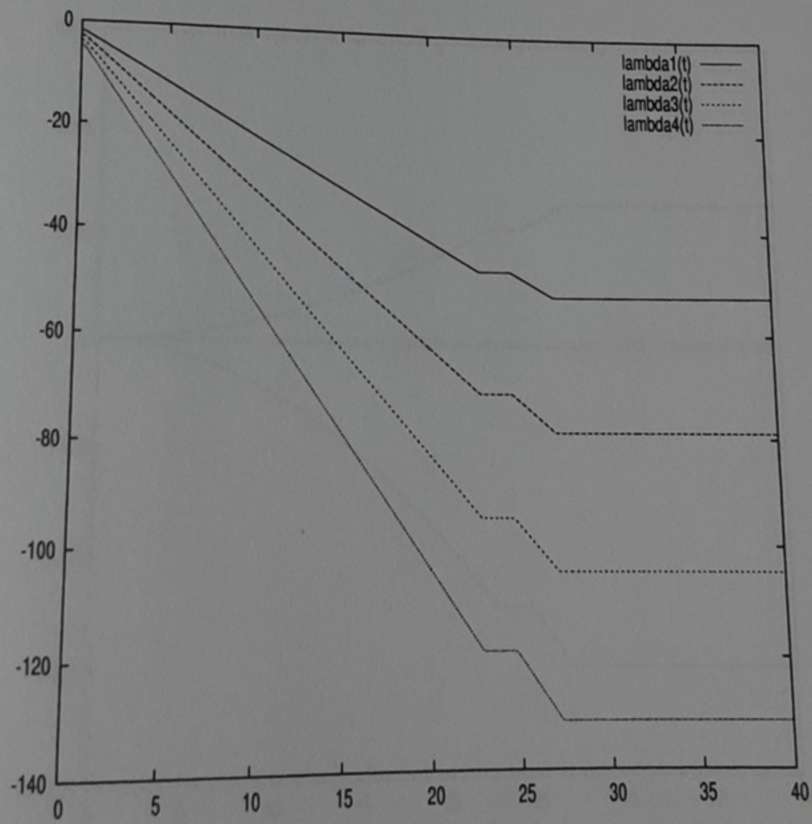

Figure 3.2.15: Histories of the eigenvalues of the error system: $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, and $\lambda_4(t)$.

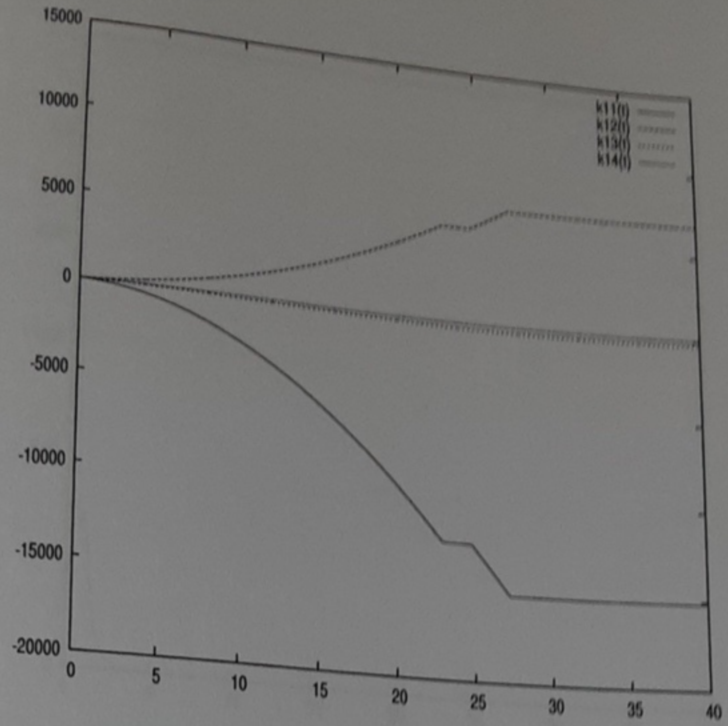

Figure 3.2.16: Histories of the feedback gains of the observer-like system: $k_{11}(t)$, $k_{12}(t)$, $k_{13}(t)$, and $k_{14}(t)$.

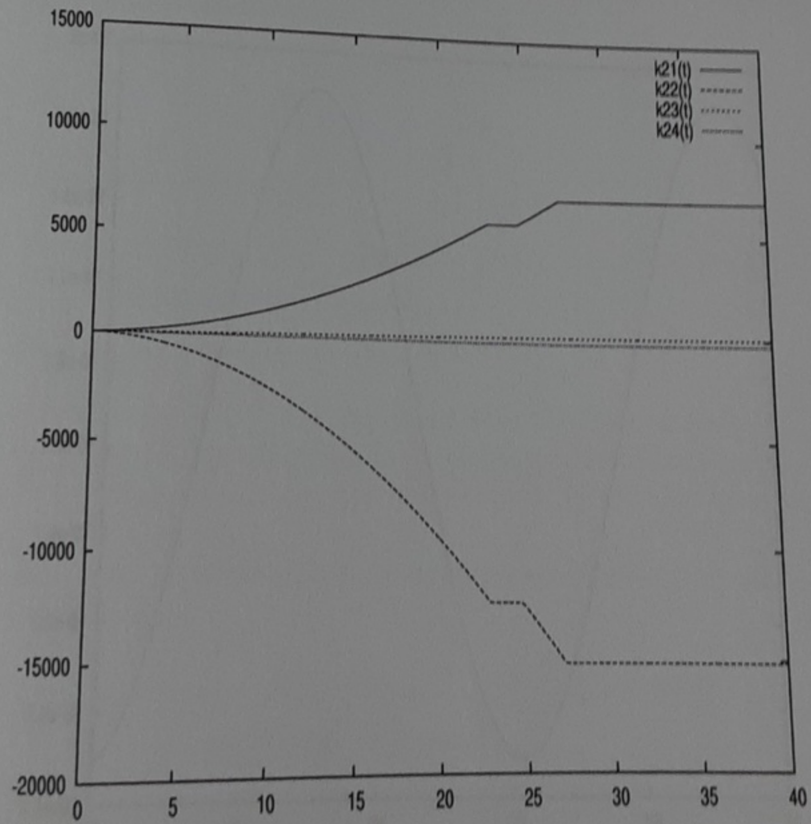

Figure 3.2.17: Histories of the feedback gains of the observer-like system: $k_{21}(t)$, $k_{22}(t)$, $k_{23}(t)$, and $k_{24}(t)$.

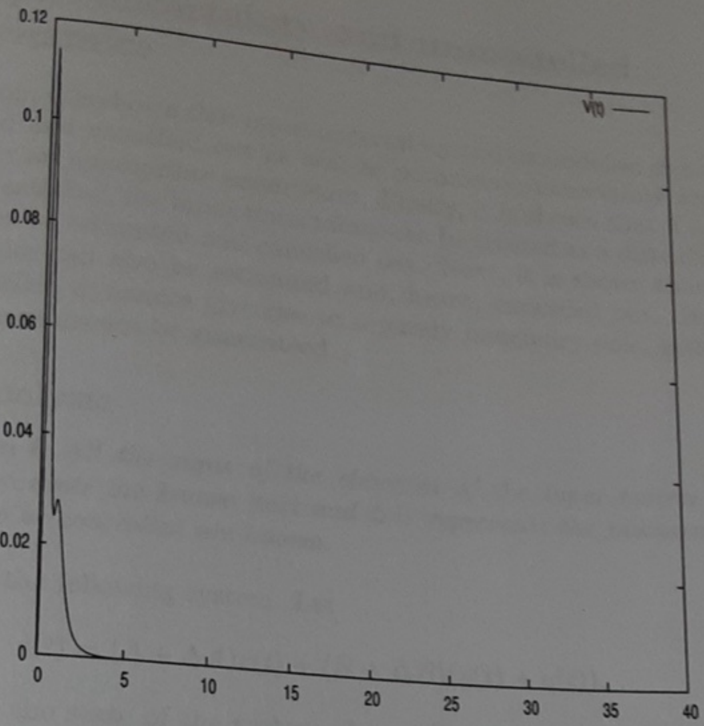

Figure 3.2.18: History of the Lyapunov-like function $V(t)$.

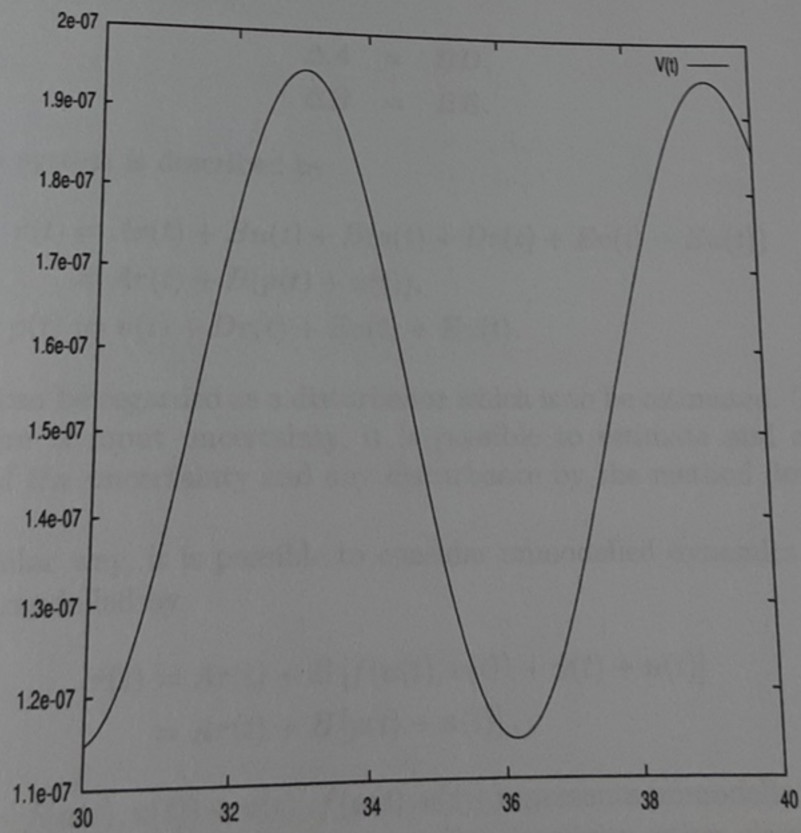

Figure 3.2.19: History of the Lyapunov-like function $V(t)$ at some later time.

3.3 Input uncertainty and unmodelled dynamics

In this section, it is shown that input uncertainty and unmodelled dynamics can be estimated and cancelled out as well as parametric uncertainty and disturbances under an appropriate assumption. Firstly, it is shown that if a matched condition is satisfied, the input uncertainty can be treated as a disturbance and, hence, it can be estimated and cancelled out. Next, it is shown that, unmodelled dynamics can also be estimated and, hence, cancelled out. However, if such unmodelled dynamics give rise to a purely imaginary pole, estimation of the disturbance cannot be guaranteed.

3.3.1 Analysis

Assumption 6 *All the signs of the elements of the input matrix $B + \Delta B$, where B represents the known part and ΔB represents the unknown part, for the system to be controlled are known.*

Consider the following system. Let

$$\dot{r}(t) = (A + \Delta A)r(t) + (B + \Delta B)(v(t) + u(t)),$$

where $r(t)$ is the state of the system, A and B are nominal system and input matrices, respectively, and ΔA, ΔB, and $v(t)$ represent parametric uncertainty, input uncertainty, and an external disturbance, respectively. Suppose the following 'matching' relations hold:

$$\begin{aligned} \Delta A &= BD, \\ \Delta B &= BE. \end{aligned}$$

Hence, the system is described by

$$\begin{aligned} \dot{r}(t) &= Ar(t) + Bu(t) + B(v(t) + Dr(t) + Ev(t) + Eu(t)) \\ &= Ar(t) + B(p(t) + u(t)), \\ p(t) &:= v(t) + Dr(t) + Ev(t) + Eu(t). \end{aligned}$$

Thus, $p(t)$ can be regarded as a disturbance which is to be estimated. Therefore, even if there is input uncertainty, it is possible to estimate and cancel out the effect of the uncertainty and any disturbance by the method developed in Chapter 2.

In a similar way, it is possible to consider unmodelled dynamics. Consider the system modelled by

$$\begin{aligned} \dot{r}(t) &= Ar(t) + B\left[f(u(t), v(t)) + v(t) + u(t)\right] \\ &= Ar(t) + B\left[p(t) + u(t)\right], \end{aligned}$$

where $p(t) = f(u(t), v(t)) + v(t)$, $f(u(t), v(t))$ represents unmodelled dynamics, and $v(t)$ represents external disturbance. For simplicity, other classes of uncertainty are omitted. It is clear that, as well as input uncertainty, it is possible to estimate and cancel out the effect of this class of uncertainty, since it can be treated as a disturbance even though it is a function of the control input to the system. A particular instance of this type of uncertainty is described by

$$f(u(t), v(t)) = \mathcal{L}^{-1}\left\{ \frac{1}{(s+\alpha)(s+\beta)} (U(s) + V(s)) \right\},$$

where $U(s) = \mathcal{L}\{u(t)\}$ and $V(s) = \mathcal{L}\{v(t)\}$. Typically, such the unmodelled dynamics can have a pole at high frequency region.

Theorem 8 *Under Assumptions 1-3 and 6, the system (2.2.5) is robust, using the disturbance estimation/cancellation method and adaptive Algorithm 4, with respect to parametric uncertainty, input uncertainty, unmodelled dynamics and external disturbances in the sense of Definition 23.*

Proof By Lemma 24, $\lambda_i(t)$ are uniformly bounded and, hence, $||K(t)||$ and $||T(t)||$ are uniformly bounded. By Lemma 10, $||\hat{e}(t)||$ is uniformly bounded. Hence, $||e(t)|| \leq ||T(t)|| \, ||\hat{e}(t)||$ is also uniformly bounded. Therefore, $||u(t)|| \leq ||K(t)|| \, ||e(t)||$ and $||f(u(t), v(t))||$, where $f(u(t), v(t))$ is unmodelled dynamics whose poles are not purely imaginary value and $v(t)$ is bounded external disturbance to the system (2.2.5), are uniformly bounded. Rest of proof is similar to the proof of Theorem 6. ■

Remark 31 In this study, it is assumed that the disturbance to the system is bounded. If unmodelled dynamics have a purely imaginary pole and if the unmodelled dynamics gives rise to reasonance, then the disturbance to the system is not bounded. Therefore, for such a situation, the estimation of the disturbance cannot be guaranteed.

3.3.2 Simulation example

In this subsection, the method of disturbance estimation and cancellation in the presence of input uncertainty and unmodelled dynamics is demonstrated by numerical simulation.

Configuration

The system, to be examined, is a second order single-input linear system expressed as follows:

$$\dot{r}(t) = Ar(t) + B(u(t) + d(t)),$$

where $d(t)$ represents external disturbance/uncertainty, and the system matrices are given by

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

An initial condition for the system is taken to be $r(t_0) = [3\ 0]^t$. For this problem, the tuning parameter of the adaptive Algorithm 4, is chosen to be $\epsilon_a^2 = 2.0 \times 10^{-6}$. For the adaptive algorithm, $\delta = -2$, $\kappa_2 = 5$, and $\omega = 10$ are used and initially, $\dot{\lambda}_1(t_0) = 0$ and $\lambda_1(t_0) = -2$ are set. For simulation purposes, the disturbance term is chosen to be $d(t) = (1+E)\{v(t) + f(u(t), v(t))\} + Eu(t)$, where

$v(t) = \sin(50t)$, $f(u(t), v(t)) = \mathcal{L}^{-1}\left\{\frac{10}{s^2+0.2s+2500.01}(U(s)+V(s))\right\}$, $U(s) = \mathcal{L}\{u(t)\}$, $V(s) = \mathcal{L}\{v(t)\}$, and $E = 2$, which represents the effect of input uncertainty. Also, an estimated disturbance is used to cancel out effect of disturbance to the system; i.e. the opposite sign of the estimated disturbance is fed back to the system. The simulation has been performed with the following configuration:

Programing language: C++;

Compiler: g++ version 3.2.2;

Algorithm to obtain numerical solution of ODE: Runge-Kutta (see [31]);

Time step for Runge-Kutta algorithm: 1.0×10^{-5}.

Simulation results

The open-loop response of the states of the system are shown in Figures 3.3.1 and 3.3.2. By these figures, it is clear that the amplitude of the states are getting larger. This is due to the fact that the imaginary parts of the poles of the unmodelled dynamics and the frequency of the external disturbance are the same and the magnitudes of the real parts of the poles of the unmodelled dynamics are relatively small.

The closed-loop response of the states of the system are shown in Figures 3.3.3 and 3.3.4. In these figures, it is observed that, unlike the open-loop system, the states converges to their equilibrium despite the existence of uncertainty and disturbance.

The error between the actual and estimated disturbances is shown in Figure 3.3.5. In this figure, it is observed that this error decreases very rapidly. The error between the actual and estimated disturbance at some later time interval is illustrated in Figure 3.3.6. In this figure, it is observed that the estimation error is small. Thus, the figure confirms that the estimation of the disturbance has been succeeded.

The histories of the feedback gains to the observer-like system and the eigenvalues of the error system are shown in Figures 3.3.7 and 3.3.8. In these figures, it is observed that these values are increased or decreased until they reach to the certain values and thereafter they remain their. The history of the Lyapunov-like function is shown in Figure 3.3.9. In this figure, it is observed that the value of this function decreases very rapidly. The history of the Lyapunov-like function at some later time period is shown in Figure 3.3.10. In this figure, it is observed that the the value of this function is bounded by the specified constant, that is $V(t) \le \epsilon_e^2 = 2.0 \times 10^{-6}$. Therefore, the simulation results confirm the analysis in the presence of the input uncertainty and unmodelled dynamics.

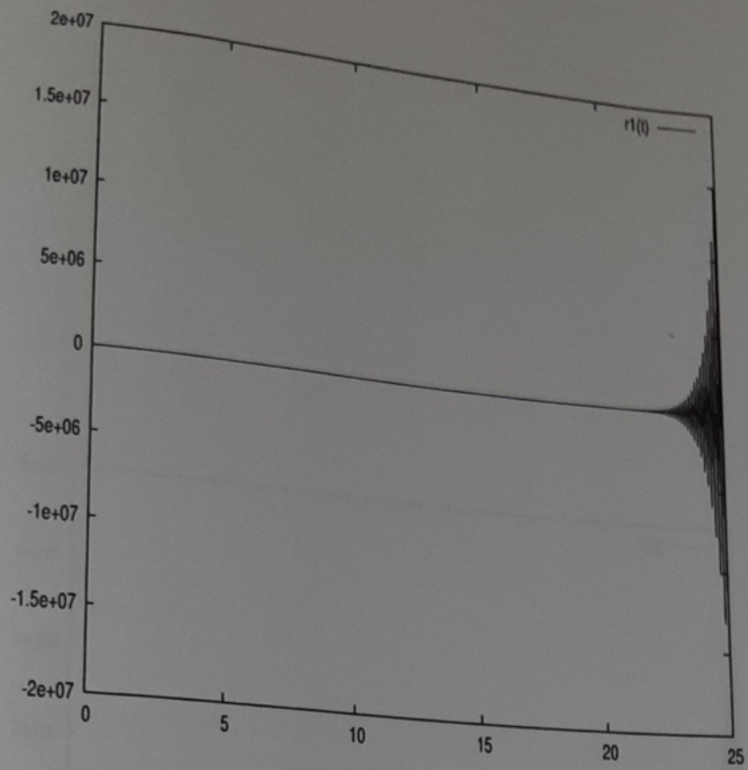

Figure 3.3.1: Open-loop response of the state $r_1(t)$.

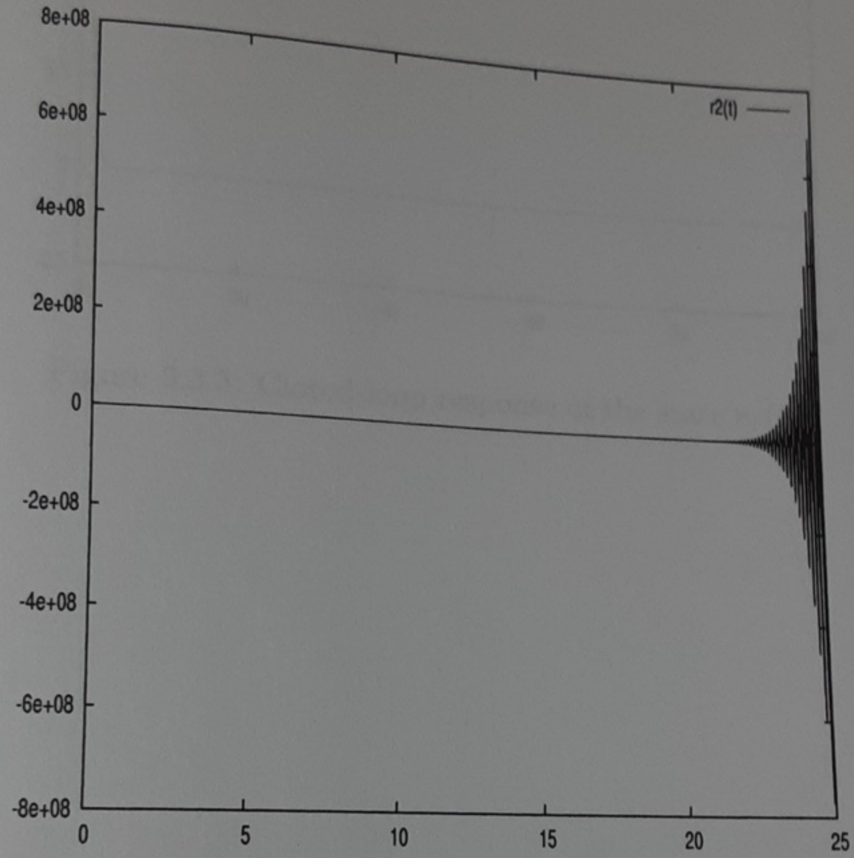

Figure 3.3.2: Open-loop response of the state $r_2(t)$.

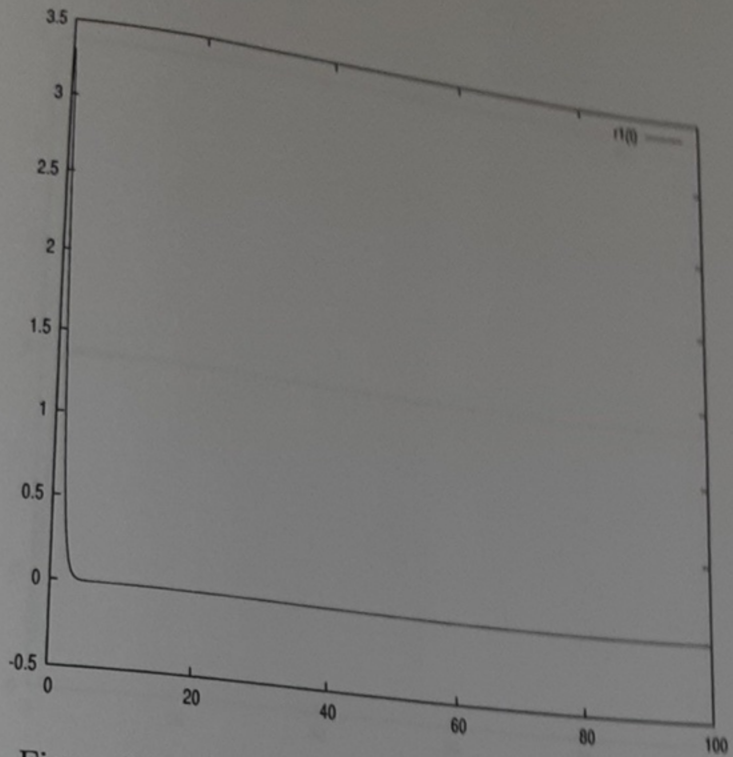

Figure 3.3.3: Closed-loop response of the state $r_1(t)$.

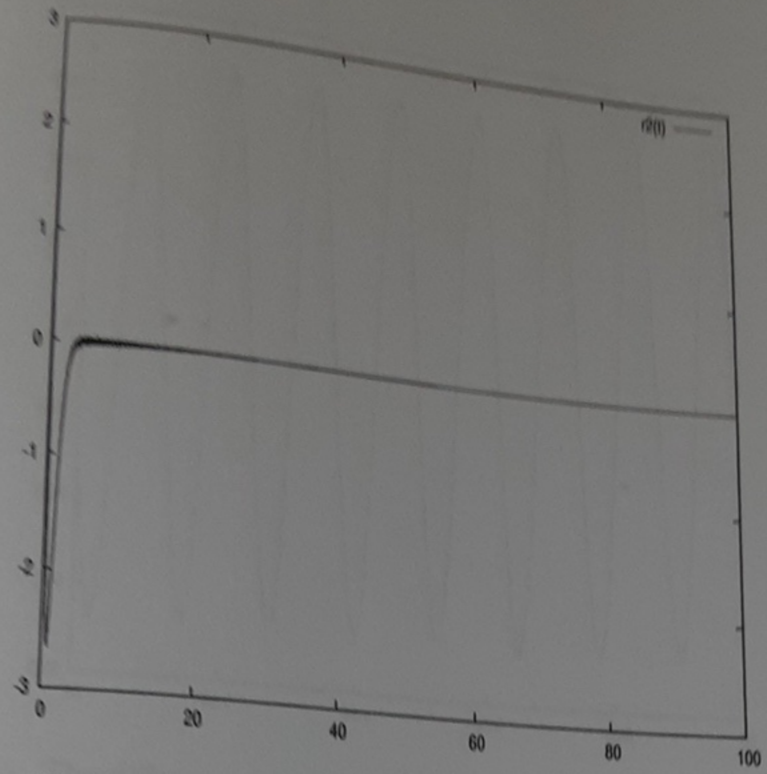

Figure 3.3.4: Closed-loop response of the state $r_2(t)$.

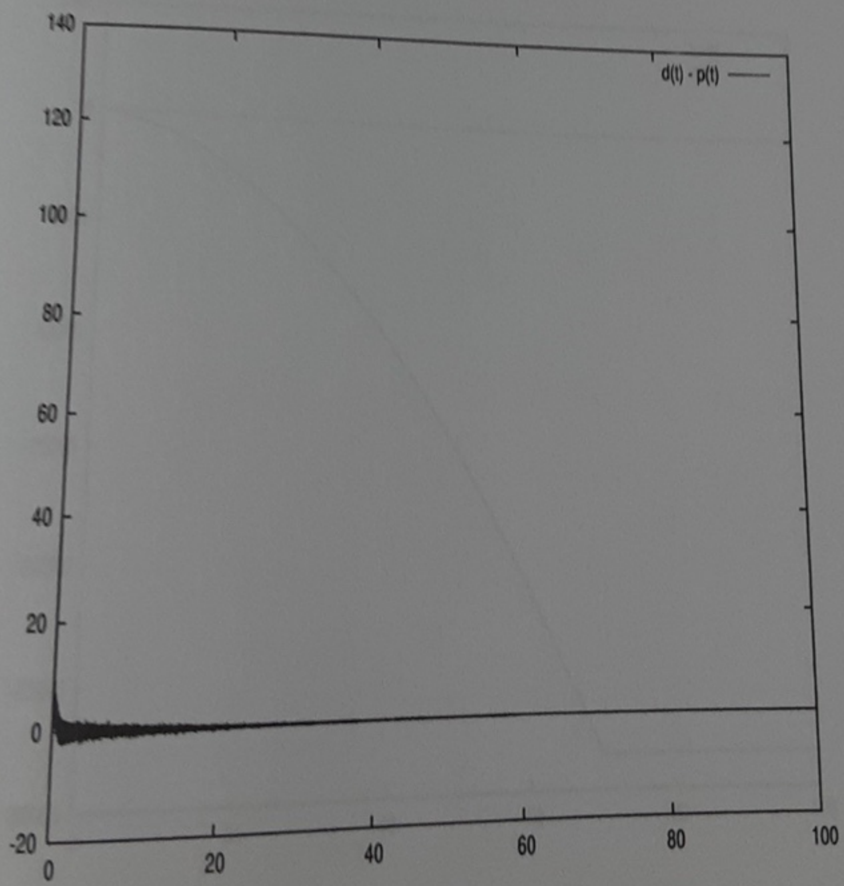

Figure 3.3.5: The error between actual and estimated disturbances, $d(t) - p(t)$.

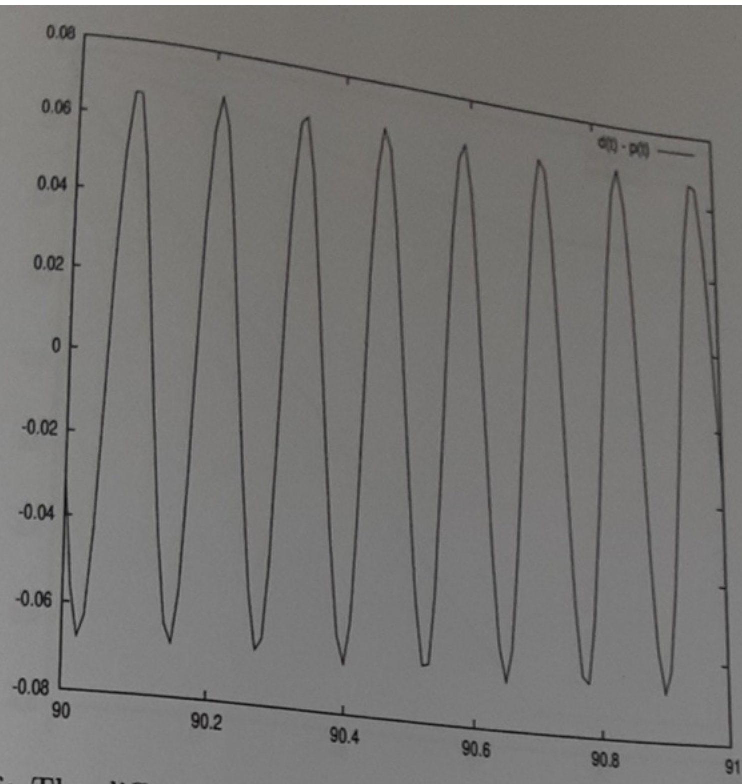

Figure 3.3.6: The difference between the actual and estimated disturbances at some later time interval.

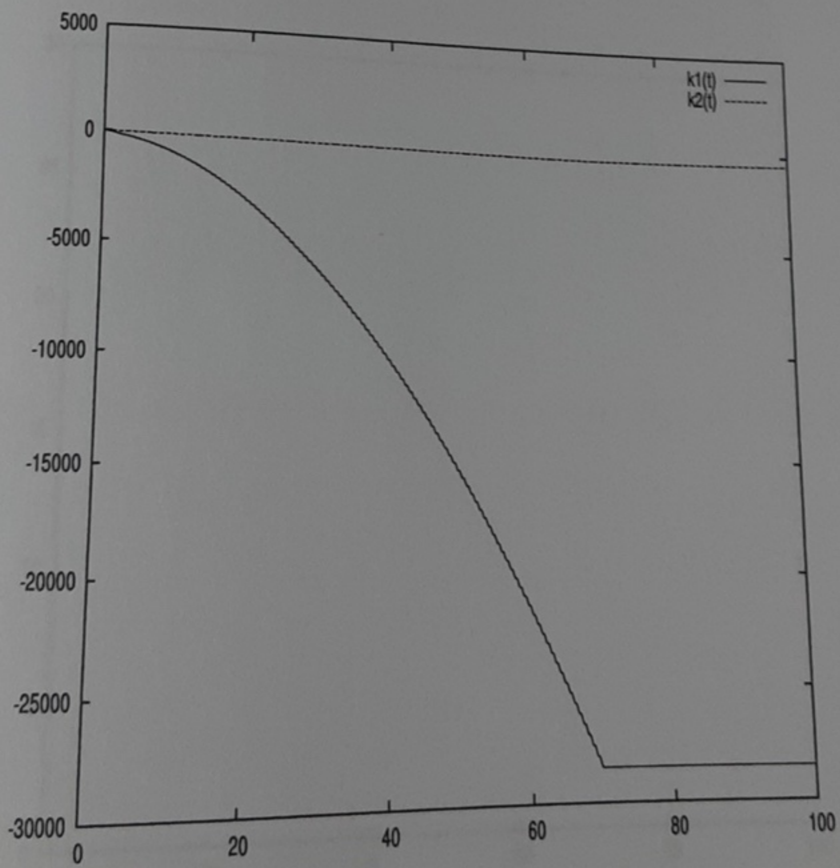

Figure 3.3.7: Histories of the feedback gains of the observer-like system: $k_1(t)$ and $k_2(t)$.

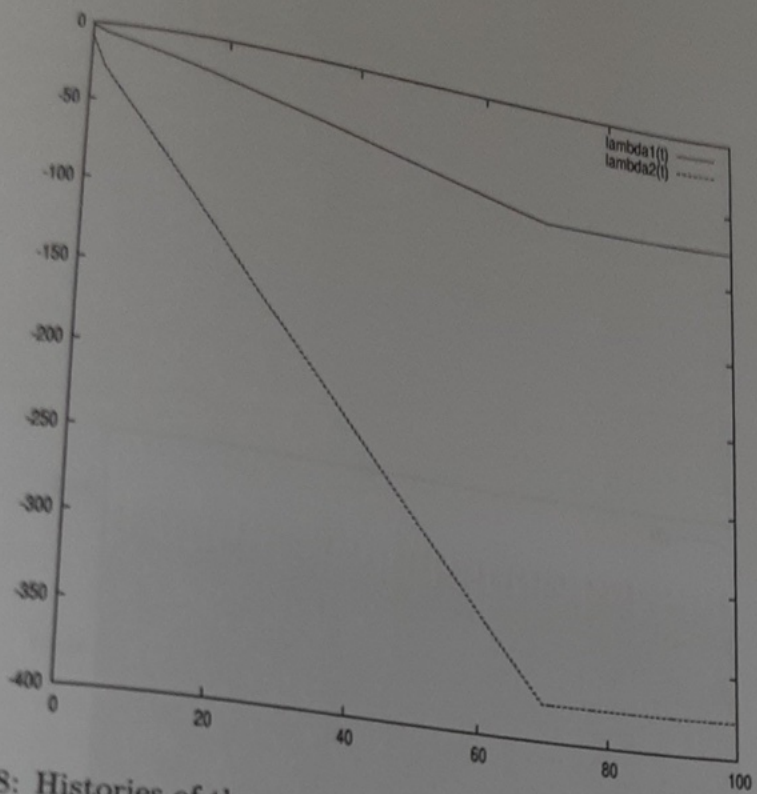

Figure 3.3.8: Histories of the eigenvalues of the error system: $\lambda_1(t)$ and $\lambda_2(t)$.

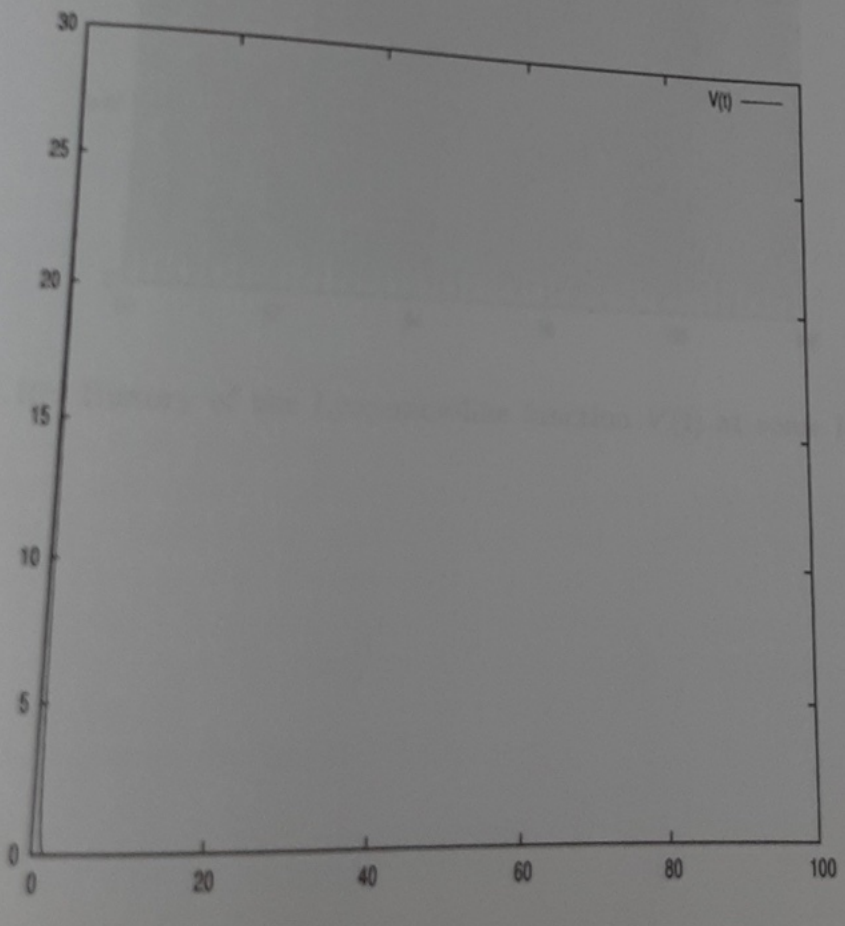

Figure 3.3.9: History of the Lyapunov-like function $V(t)$.

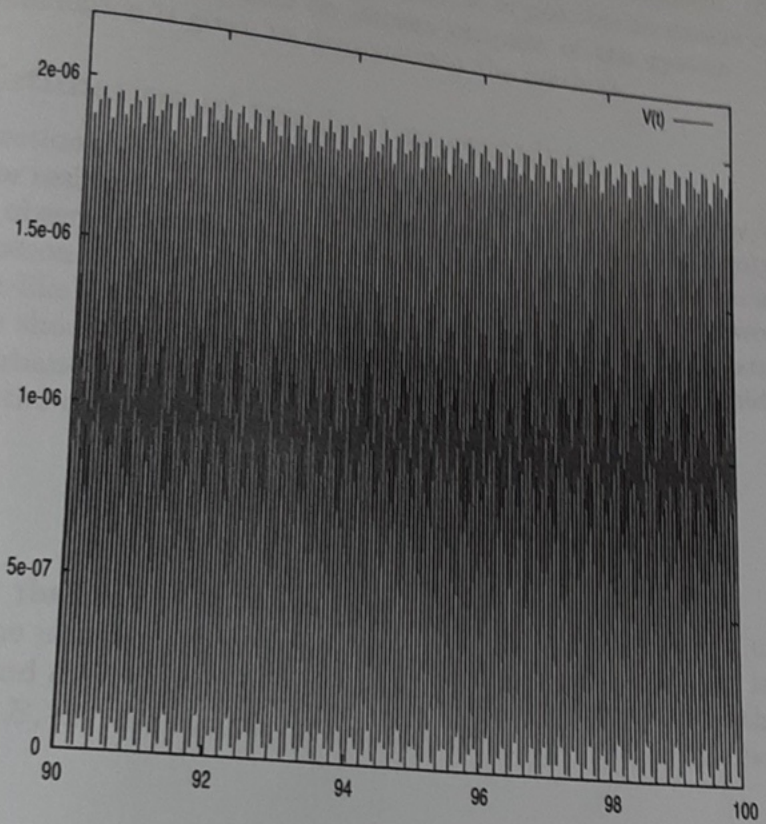

Figure 3.3.10: History of the Lyapunov-like function $V(t)$ at some later time interval.

3.4 Residual uncertainty

In this section, it is shown that it is always possible to estimate any residual uncertainty/disturbance and it is possible to cancel out the effect of residual uncertainty/disturbance to a system under appropriate conditions. The method of estimation is based on the fact that the input matrix for the observer-like system, used for estimation, can be modified by the control designer. The method of cancellation is based on feedforward cancellation of the uncertainty/disturbance using the estimated uncertainty/disturbance. The outline of this section is given as follows. Firstly, it is shown that it is always possible to estimate any effect of uncertainty/disturbance using a modified observer-like system. Next, it is shown that under an appropriate condition, it is possible to cancel out the effect of uncertainty/disturbance on chosen outputs of the system. Finally, a numerical simulation is given to demonstrate the method.

3.4.1 Estimation of residual uncertainty

In this subsection, it is shown that regardless whether the uncertainty/disturbance is matched or residual, it is always possible to estimate such uncertainty/disturbance by using an observer-like system. The basic framework of estimation is the same as for estimation of matched uncertainty. The main point, in this work, is that the observer-like system can be altered by modifying the input matrix to that system. It is shown that using this characteristics, estimation of residual uncertainty/disturbance is possible.

Suppose the open-loop system is

$$\begin{aligned} \dot{r}(t) &= (A + \Delta A)r(t) + v(t) \\ &= Ar(t) + \Delta Ar(t) + v(t) \end{aligned} \tag{3.4.1}$$

where $r(t)$ is the state of the system, ΔA represents parametric uncertainty, and $v(t)$ is the external disturbance. For simplicity, the situation in which all uncertainty and disturbance do not satisfy a matched condition with respect to input matrix B, for the system, is considered. Thus, (3.4.1) can be expressed in the form:

$$\dot{r}(t) = Ar(t) + w(t),$$

where $w(t) = \Delta Ar(t) + v(t)$ is residual uncertainty.

When estimation/cancellation of matched uncertainty/disturbance is performed using the observer-like system (2.2.6), the input matrix of the observer-like system is the same as the one for the system (2.2.5). As a result, only matched uncertainty/disturbance can be estimated. However, since the observer-like system can be altered by modifying the input matrix, estimation of residual uncertainty is possible.

Define the observer-like system to be:

$$\dot{x}(t) = Ax(t) + \bar{B}\bar{u}(t).$$

Note that the input matrix for this system, $\bar{B}$, is different from B. It is supposed that the disturbance/uncertainty of the system, $w(t)$, satisfies a matched condition with respect to $\bar{B}$, that is there exists $p(t)$ such that $\bar{B}p(t) = w(t)$.

As a result of the above formulations, the system and the observer-like system can be expressed as

$$\begin{aligned}\dot{r}(t) &= Ar(t) + w(t) \\ &= Ar(t) + \bar{B}p(t), \\ \dot{x}(t) &= Ax(t) + \bar{B}\bar{u}(t).\end{aligned}$$

Since they both have exactly the same structure as the system described in Section 2.2, disturbance estimation can be achieved by using the modified observer-like system, defined above.

A typical example of such a problem is given next.

Example 1 Consider the following system:

$$\begin{bmatrix}\dot{r}_1(t)\\ \dot{r}_2(t)\\ \dot{r}_3(t)\\ \dot{r}_4(t)\\ \dot{r}_5(t)\\ \dot{r}_6(t)\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46}\\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56}\\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}\end{bmatrix}\begin{bmatrix}r_1(t)\\ r_2(t)\\ r_3(t)\\ r_4(t)\\ r_5(t)\\ r_6(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ v_1\end{bmatrix}$$

where $r_i(t)$ $(i = 1, \cdots, 6)$ are the states of the system, $u_j(t)$ $(j = 1, 2)$ are the control inputs and $v_1(t)$ is the uncertainty/disturbance. Clearly, it is impossible to estimate all the uncertainty/disturbance if the input matrix of the observer-like system is the same as the input matrix for the system. However, it is possible to estimate all residual uncertainty/disturbance by using an appropriate input matrix. Define the input matrix, $\bar{B}$, and inputs of the observer-like system, $\bar{u}(t)$, as follows:

$$\bar{B} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

$$\bar{u}(t) = \begin{bmatrix}\bar{u}_1(t)\\ \bar{u}_2(t)\\ \bar{u}_3(t)\end{bmatrix}.$$

It is clear that using this input matrix, all the residual uncertainty/disturbance can be treated as matched uncertainty/disturbance. Therefore, it is possible to estimate any uncertainty/disturbance, regardless of whether it is matched or residual.

3.4.2 Exact cancellation of residual uncertainty

In this subsection, the method of exact cancellation of residual uncertainty and/or disturbance is introduced. The method is based on feedforward control with respect to the estimated disturbance. Firstly, a description of the problem is given. Next, it is shown that it is always possible to construct a control input vector which cancels out the effect of uncertainty/disturbance affecting appropriate outputs.

Consider the following system:

$$\dot{r}(t) = Ar(t) + Bu(t) + \bar{B}p(t) \quad (3.4.2)$$
$$y(t) = Cr(t) \quad (3.4.3)$$

where $r(t) \in \mathbb{R}^n$ is the state of the system, $y(t) \in \mathbb{R}^p$ is a vector of the chosen outputs of the system, $u(t) \in \mathbb{R}^m$ $(m \geq p)$ is the control input to the system, and $p(t) \in \mathbb{R}^l$ is matched and residual uncertainty/disturbance to the system. The objective is to cancel out the effect of the residual uncertainty/disturbance to certain outputs of the system. In general, it is impossible to cancel out all the effects of the residual uncertainty/disturbance to every state of a system. However, in practice, it might be sufficient to cancel out residual uncertainty/disturbance for some chosen outputs or states of a system if the number of chosen outputs or states is greater than the number of inputs. For example, consider the system described by

$$\begin{bmatrix} \dot{r}_1(t) \\ \dot{r}_2(t) \\ \dot{r}_3(t) \\ \dot{r}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \\ r_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} p(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \\ \dot{r}_1(t) \\ \dot{r}_2(t) \end{bmatrix},$$

where $r_3(t) = \dot{r}_1(t)$, $r_4(t) = \dot{r}_2(t)$, $p(t)$ is residual uncertainty/disturbance and the objective of the control is to control $r_1(t)$, or $y(t)$, by $u(t)$. For this system, the number of outputs to be controlled and the number of inputs for control is the same. Thus, the problem of cancellation of residual uncertainty/disturbance to p outputs and m inputs is considered when $p \leq m$. In the following description, the objective is to control $y(t)$ for the system described by (3.4.2) and (3.4.3). Consider the uncontrolled system

$$\dot{r}(t) = Ar(t) + \bar{B}p(t), \quad (3.4.4)$$
$$y(t) = Cr(t). \quad (3.4.5)$$

The input-output relation of this system is given by

$$Y(s) = G(s)P(s),$$

where $Y(s) = \mathcal{L}\{y(t)\}$, $P(s) = \mathcal{L}\{p(t)\}$, and $G(s) = C(sI - A)^{-1}\bar{B}$. In order to design $u(t)$ to cancel out the uncertainty, consider the system:

$$\dot{r}(t) = Ar(t) + B\bar{u}(t), \quad (3.4.6)$$
$$y(t) = Cr(t), \quad (3.4.7)$$

where $\bar{u}(t)$ is some input to the system. The input-output relation of this system is given by

$$Y(s) = \bar{G}(s)\bar{U}(s),$$

where $Y(s) = \mathcal{L}\{y(t)\}$, $F(s) = \mathcal{L}\{d(t)\}$, and $G(s) = C(sI - A)^{-1}\bar{B}$. When the output to systems (3.4.6)-(3.4.5) is approximately the same as the output to systems (3.4.6)-(3.4.7) for t sufficiently large, then

$$\tilde{G}(s)\tilde{U}(s) \approx G(s)F(s).$$

Hence,

$$\tilde{U}(s) \approx \tilde{G}^{-1}(s)G(s)F(s).$$

In order to cancel out effect of $F(s)$, it is required to design $U(s)$ so that

$$\begin{aligned} U(s) &= -\tilde{U}(s) \\ &\approx -\tilde{G}^{-1}(s)G(s)F(s). \end{aligned} \tag{3.4.8}$$

In Subsection 3.4.1, it has already been shown that, regardless of matched condition of uncertainty/disturbance with respect to input matrix of the system, it is always possible to estimate residual uncertainty/disturbance. In other words, almost exact knowledge of $p(t)$ is available. Thus, using (3.4.8), and the estimate of $p(t)$, the effect of disturbance/uncertainty to outputs can be cancelled out if the number of control inputs is greater than or equal to the number of outputs to be controlled.

3.4.3 Simulation example

Configuration

The system, to be examined, is a two-input linear system which is modelled as follows:

$$\begin{aligned} \dot{r}(t) &= Ar(t) + Bu(t) + \bar{B}d(t), \\ y_r(t) &= Cr(t), && (3.4.9) \\ u(t) &= u_r(t) + u_{tr}(t), && (3.4.10) \end{aligned}$$

where $d(t) \in \mathbb{R}^2$ is the uncertainty/disturbance, $u(t) \in \mathbb{R}$ is the control input, $u_r(t) \in \mathbb{R}$ is the control input to cancel out the effect of the uncertainty and disturbance, $u_{tr}(t) \in \mathbb{R}$ is the control input for the tracking, and the system matrices are given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & -0.1 & -5 & -0.05 \\ -0.2 & -20 & -0.1 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

An initial condition for the system is taken to be $r(t_0) = [0\ 0\ 0\ 0]^t$. The uncertainties, to be applied to the system, are given as follows:

$$\begin{aligned} d_1(t) &= \sin(t) + 2r_1(t) + 0.03r_2(t) + 1.0r_3(t) + 0.01r_4(t), \\ d_2(t) &= \sin(t + 0.1745) + 0.01r_1(t) + 3.0r_2(t) + 0.01r_3(t) + 3.0r_4(t), \end{aligned}$$

where $d_i(\cdot)$ are components of $d(t) = [d_1(t)\ d_2(t)]^t$ and $r_i(\cdot)$ are components of $r(t) = [r_1(t)\ r_2(t)\ r_3(t)\ r_4(t)]^t$. The objective of the control is to track the reference signal $R(t) = 1$ by $r_1(t)$ and, hence, $C = [1\ 0\ 0\ 0]$. The control input for the tracking, which is $u_{tr}(t)$, is designed as follows. The transfer function of system (3.4.9)-(3.4.10) between $u_{tr}(t)$ to $y_r(t)$ is given by

$$G(s) = C(sI - A)^{-1}B$$
$$= \frac{s^2 + 9s + 20}{s^4 + 14s^3 + 70.995s^2 + 153.98s + 119.98}.$$

Hence, the steady state gain of this transfer function with respect to the constant signal is given by

$$|G(0)| = \frac{20}{119.98}.$$

Thus, the control input for tracking is determined as follows:

$$u_{tr}(t) = |G(0)|^{-1}R(t)$$
$$= 5.999.$$

The following parameters are used for adaptive algorithm. $\epsilon_e^2 = 2.0 \times 10^{-6}$, $\omega = 10$, $\delta = 2.0$, $\kappa_2 = 2.0$, $\kappa_3 = 3.0$, $\kappa_4 = 4.0$, where ϵ_e represents accuracy of the estimation, ω is used to ensure that $\lambda_i(t)$ are continuous, δ is used to determine the amplitude of $\dot{\lambda}_i(t)$, and κ_i are used to determine the ratio between each eigenvalue. Also, $\lambda_1(t_0) = -2.0$ and $\dot{\lambda}_1(t_0) = 0$ are set. Also, estimated disturbances are used to cancel out the effect of disturbances to the system; i.e. the estimated disturbances are fed back to the system through the filter defined by (3.4.8). The simulation has been performed with the following configuration:

Programming language: C++;

Compiler: g++ version 3.2.2;

Algorithm to obtain numerical solution of ODE: Runge-Kutta (see [31]);

Time step for Runge-Kutta method: 1.0×10^{-5};

Algorithm to obtain inverse matrix: Gauss-Jordan Elimination (see [31]).

Simulation results

The open-loop response of the states of the system are shown at Figures 3.4.1 to 3.4.4. From these figures, it is clear that, in the presence of disturbances, these states are disturbed from their equilibrium states.

The actual and estimated disturbances are shown in Figures 3.4.10 and 3.4.11. For each figure, the solid line and the dashed line represent actual and estimated disturbances, respectively. In these figures, it is observed that the estimated disturbances converge to the actual disturbances very rapidly. The closed-loop response of the states of the system are shown in Figures 3.4.5 to 3.4.9. In Figure 3.4.5, the solid line represents the states to be regulated and the dashed line represents the reference signal. In this figure, it is clear that unlike the open-loop system, the state converges to the reference signal despite the presence of the disturbances. The state and reference signal at a later time is shown in Figure 3.4.9. In this figure, it is observed that, although there is some tracking error, that error is small. Also, note that the control input to the system, which is generated by the estimated disturbances though filter defined by (3.4.8), is shown in Figure 3.4.12. Therefore, one can conclude that the method of estimation and cancellation of residual disturbance in this section has succeeded.

The histories of the gains of the observer-like system and the eigenvalues of the error system are shown in Figures 3.4.13 to 3.4.15. In these figures, it is observed that these values are increased or decreased until they reach certain values and remain there. The history of the value of the Lyapunov-like function is shown in Figure 3.4.16. In this figure, it is observed that the value of this function decreases very rapidly. The time history of the value of the Lyapunov-like function at a later time is shown in Figure 3.4.17. In this figure, it is observed that the value of this function is bounded by a prescribed constant. Therefore, one can conclude that, for estimation of residual disturbance, the gains of the observer-like system, the eigenvalues of the error system, and the value of the Lyapunov-like function have the same properties as for the case of estimation of matched disturbance.

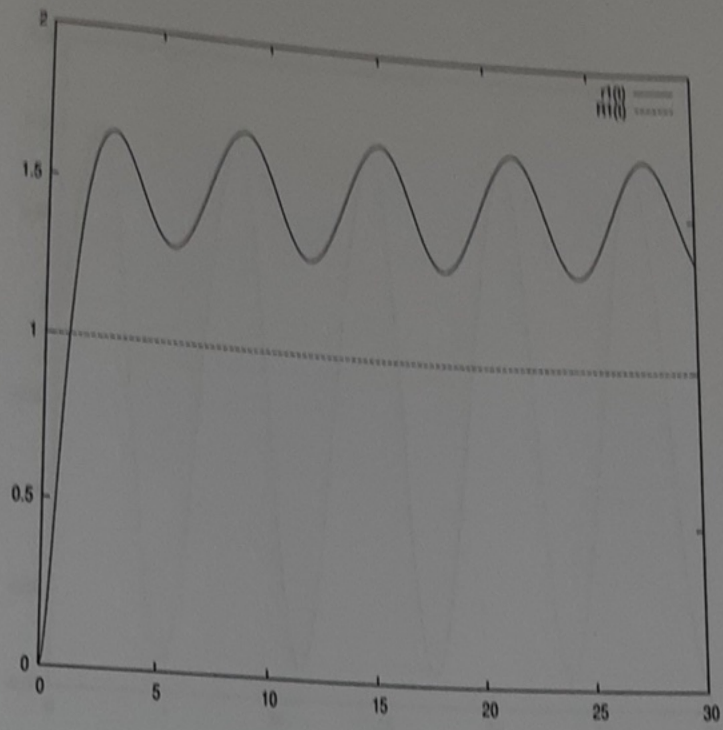

Figure 3.4.1: Open-loop response of the state $r_1(t)$.

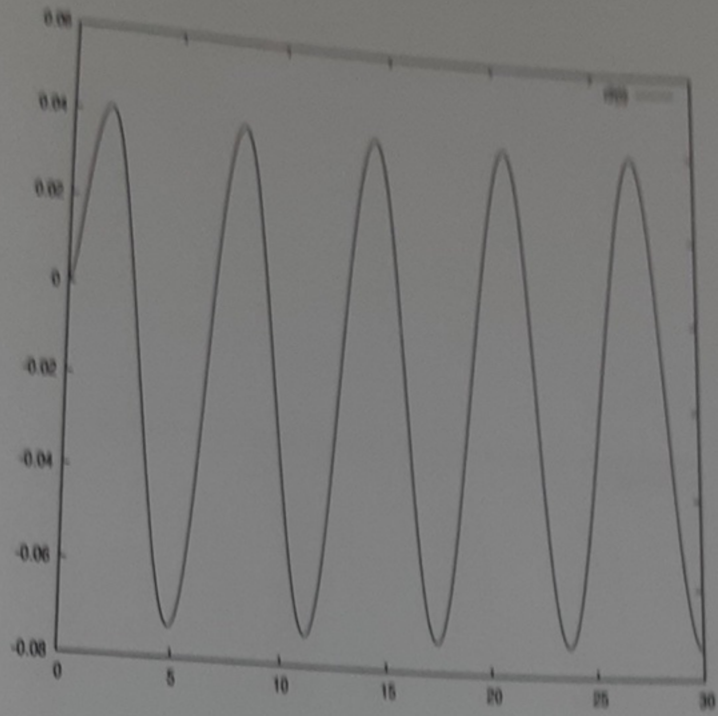

Figure 3.4.2: Open-loop response of the state $r_2(t)$.

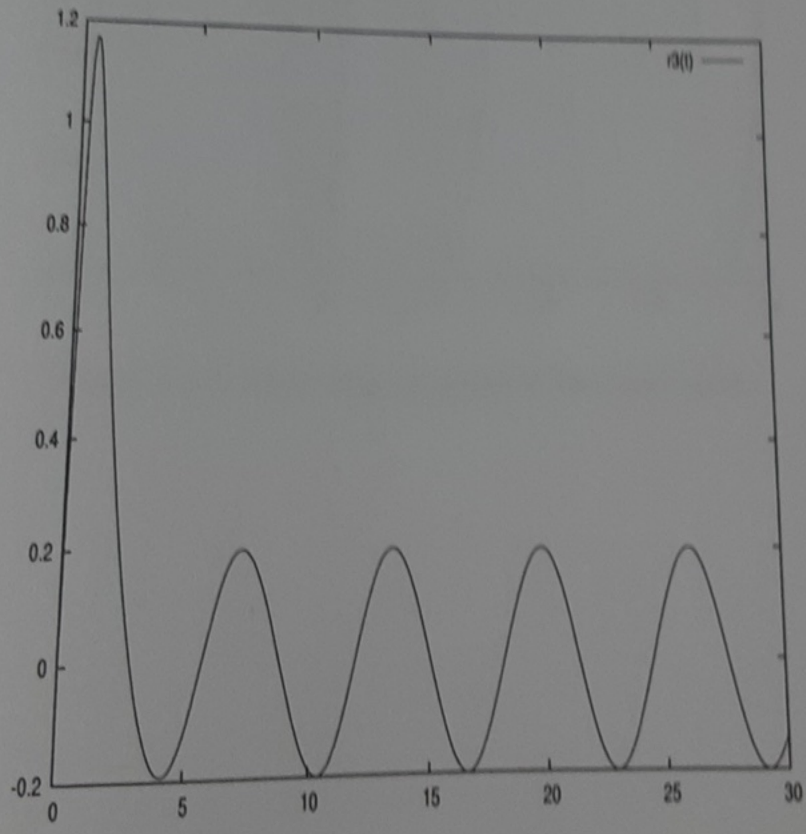

Figure 3.4.3: Open-loop response of the state $r_3(t)$.

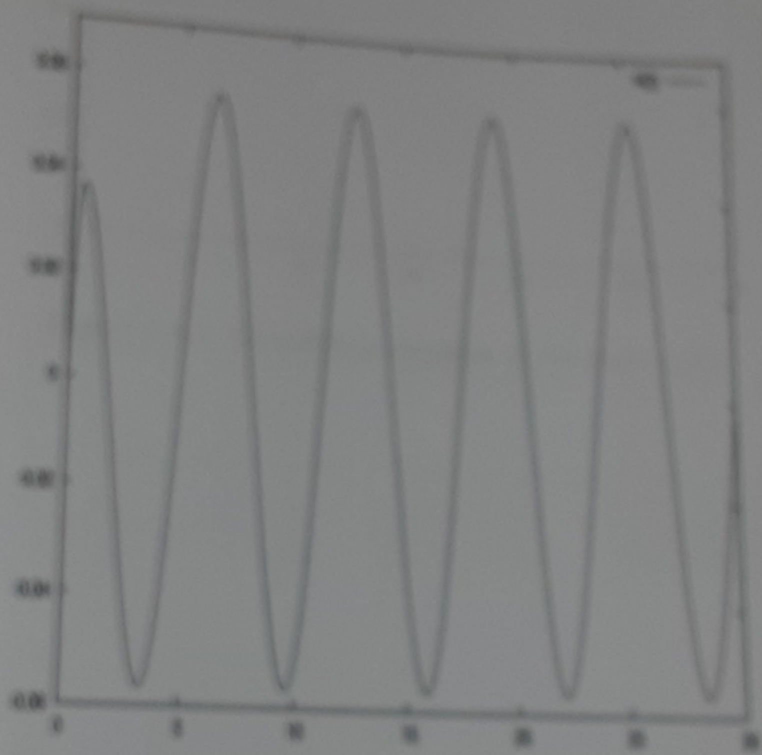

Figure 3.4.4: Open-loop response of the state $x_4(t)$.

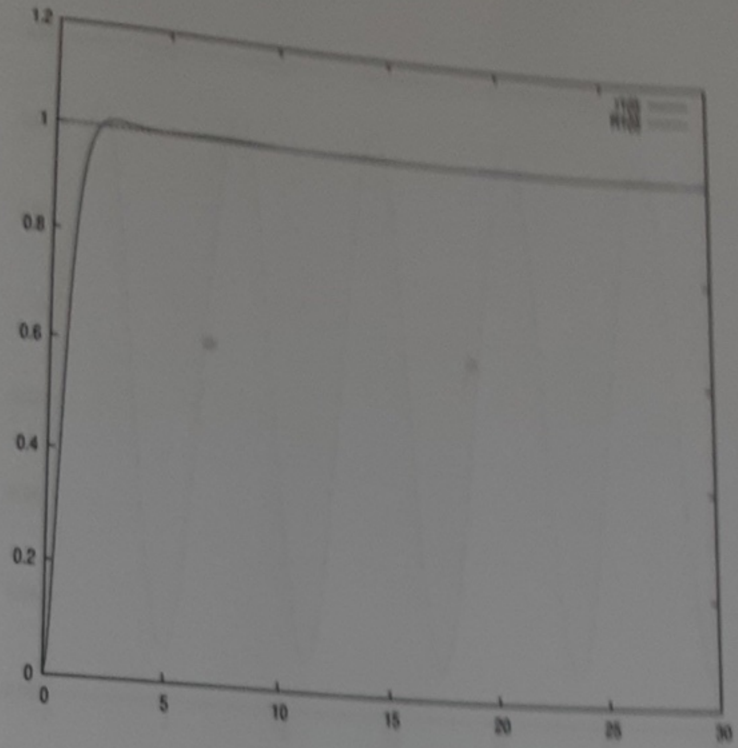

Figure 3.4.5: Closed-loop response of the state $r_1(t)$.

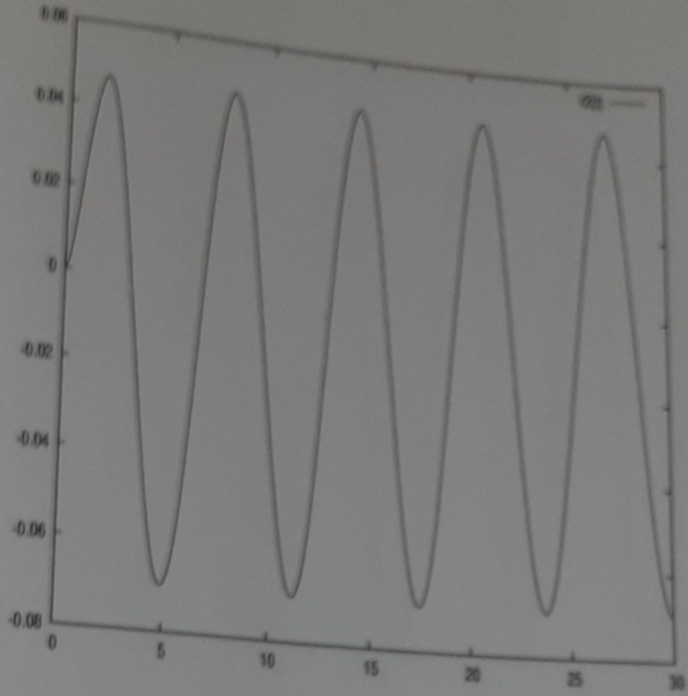

Figure 3.4.6: Closed-loop response of the state $r_2(t)$.

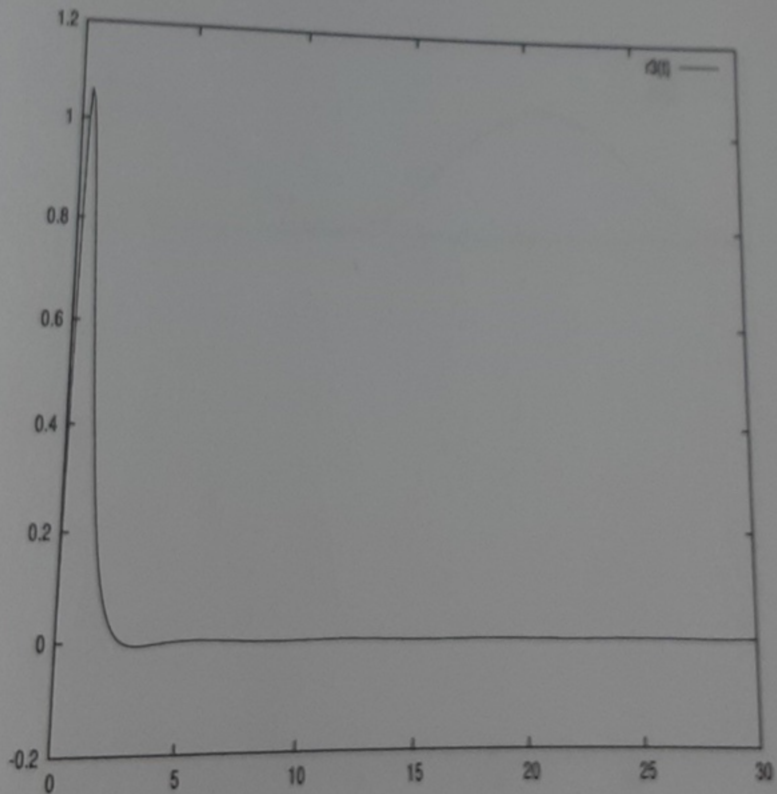

Figure 3.4.7: Closed-loop response of the state $r_3(t)$.

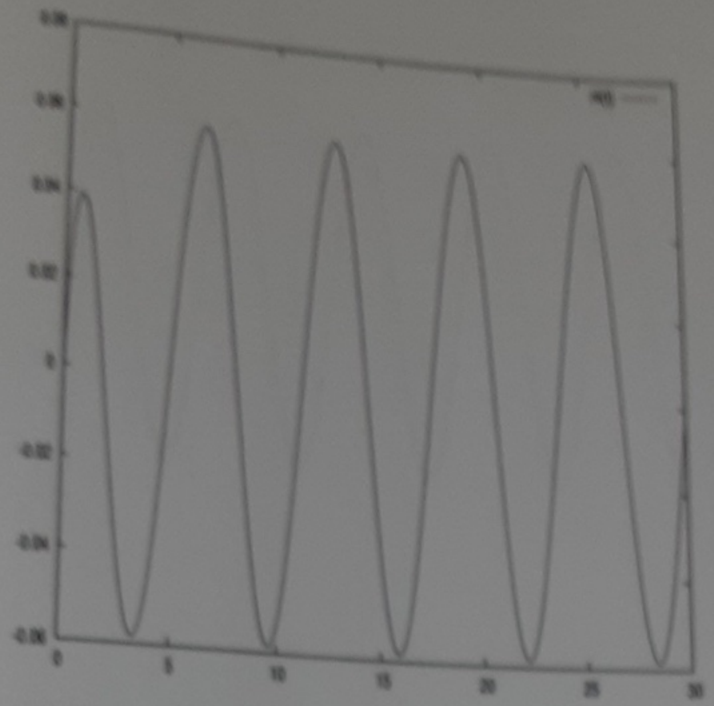

Figure 3.4.8: Closed-loop response of the state $r_4(t)$.

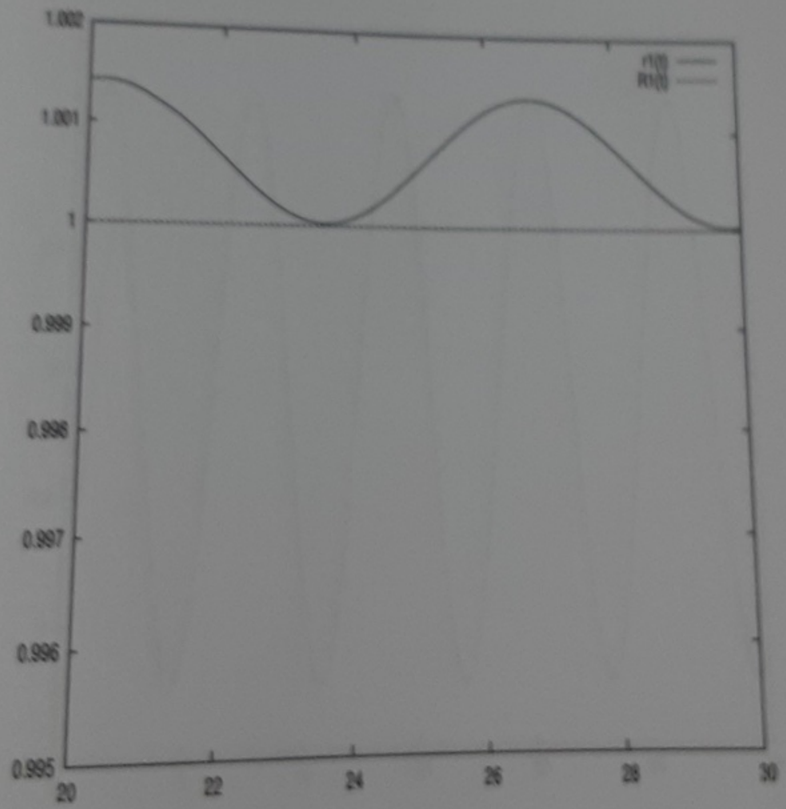

Figure 3.4.9: Closed-loop response of the state $r_1(t)$.

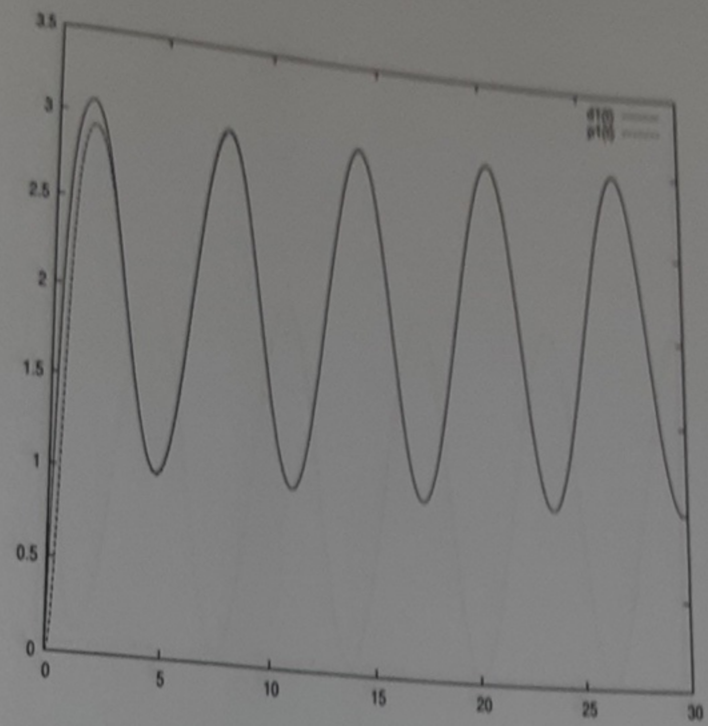

Figure 3.4.10: The actual and estimated disturbances: $d_1(t)$ and $p_1(t)$.

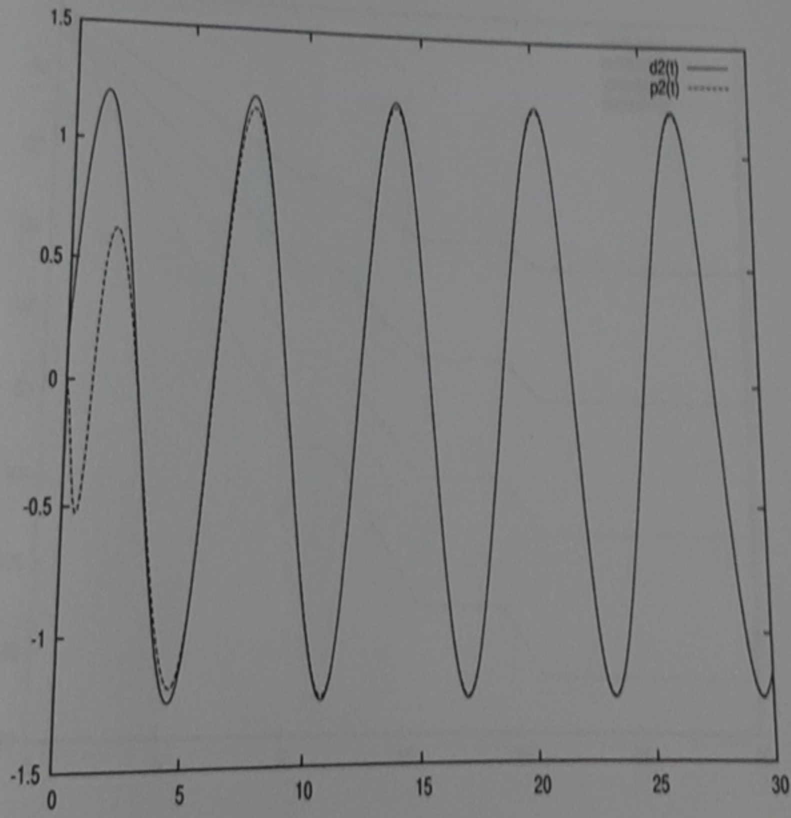

Figure 3.4.11: The actual and estimated disturbances: $d_2(t)$ and $p_2(t)$.

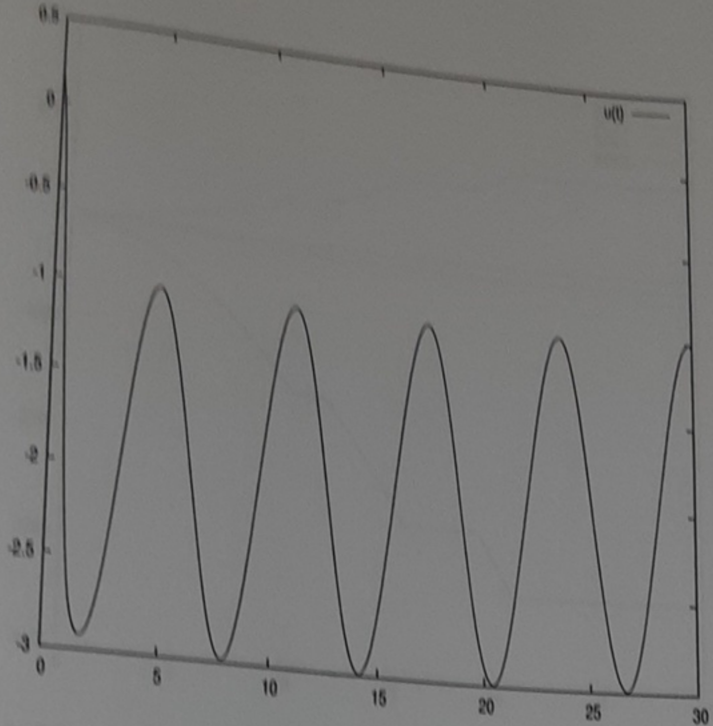

Figure 3.4.12: History of control input $u(t)$.

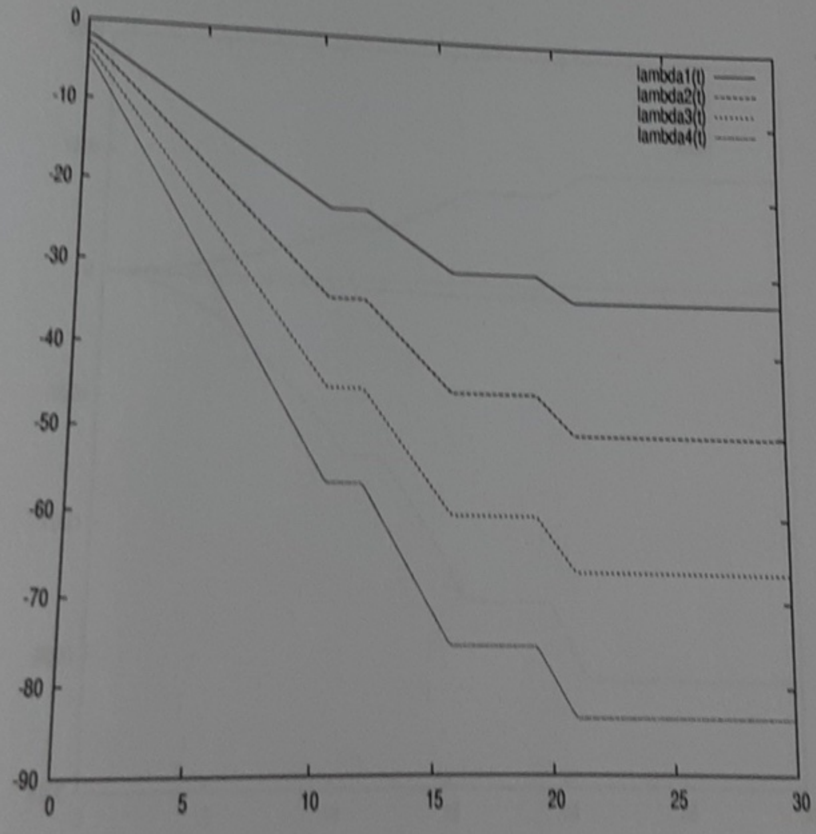

Figure 3.4.13: Histories of the eigenvalues of the error system $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, and $\lambda_4(t)$.

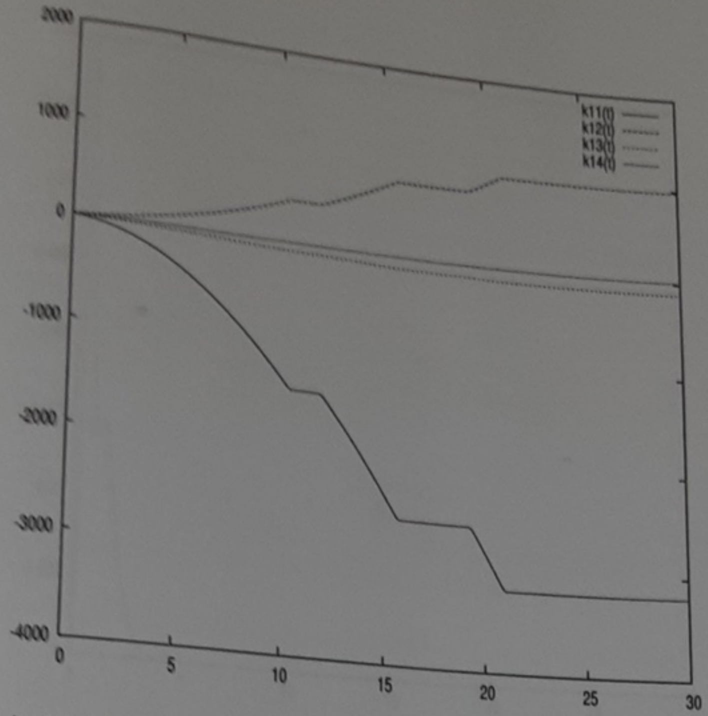

Figure 3.4.14: Histories of the feedback gains of the observer-like system: $k_{11}(t)$, $k_{12}(t)$, $k_{13}(t)$, and $k_{14}(t)$.

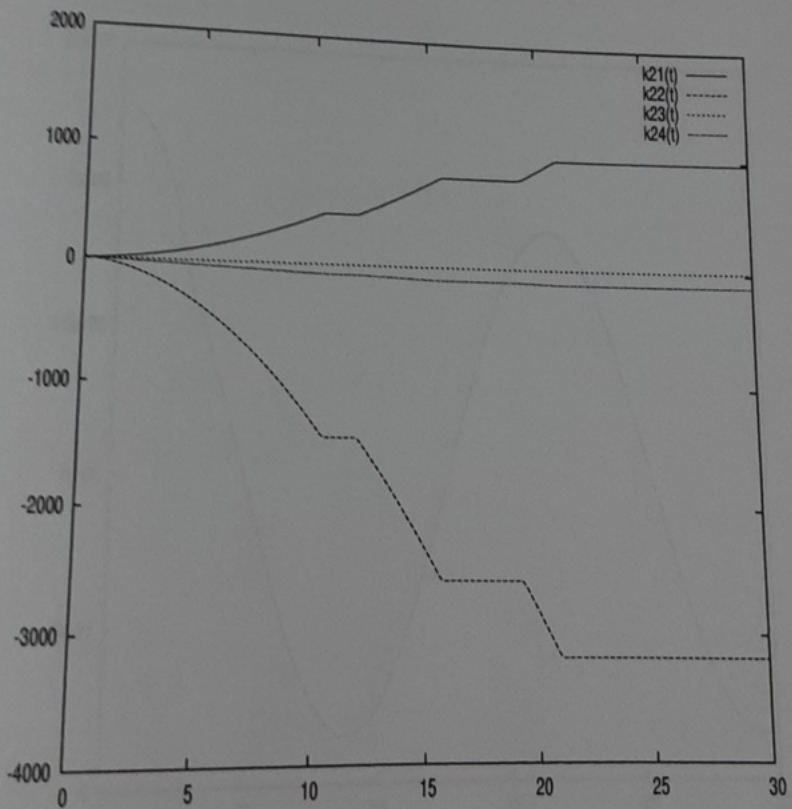

Figure 3.4.15: Histories of the feedback gains of the observer-like system $k_{21}(t)$, $k_{22}(t)$, $k_{23}(t)$, and $k_{24}(t)$.

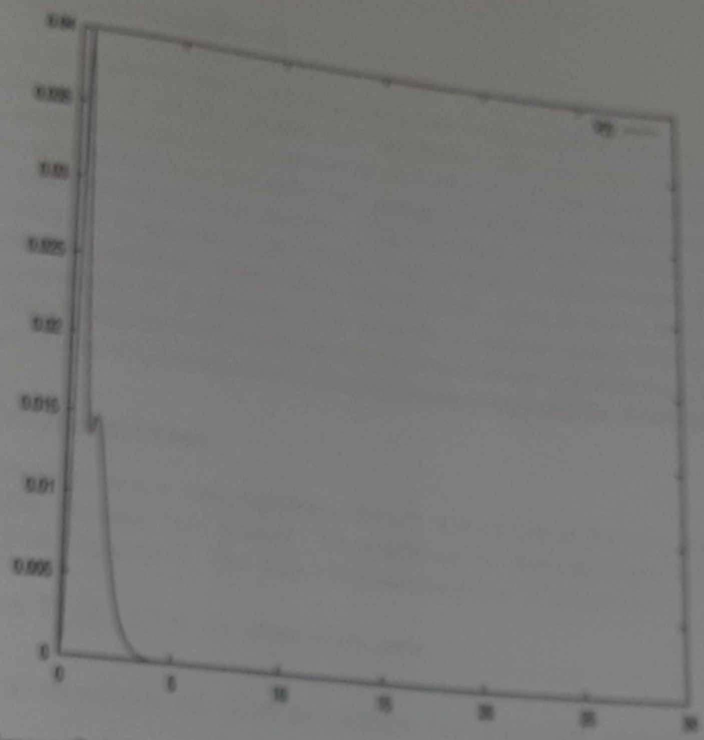

Figure 3.4.16: History of the Lyapunov-like function $V(t)$.

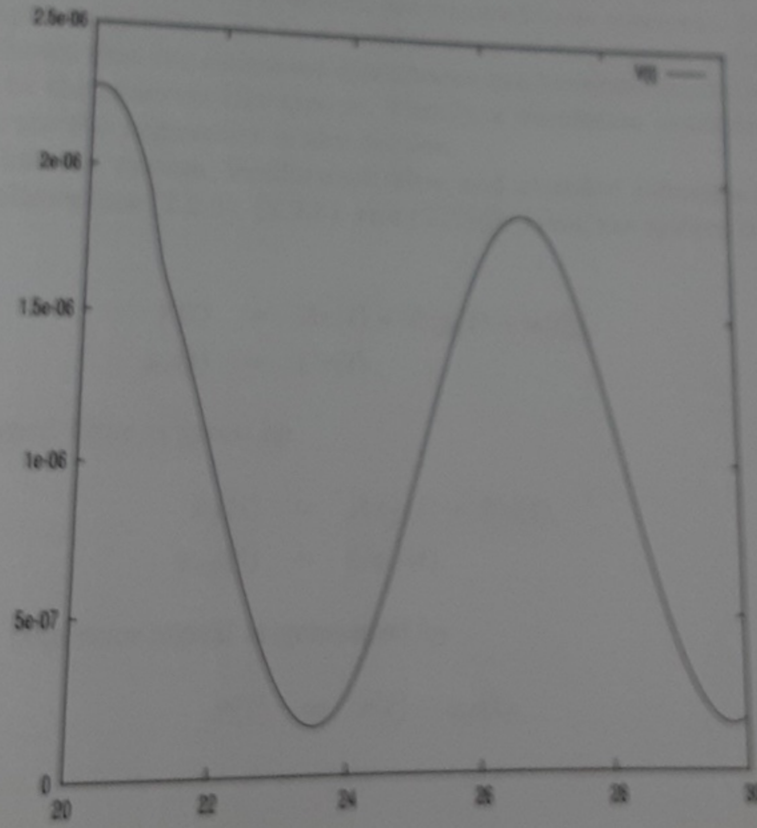

Figure 3.4.17: History of the Lyapunov-like function $V(t)$ at some later time.

3.5 Output control

In Chapter 2, it is shown that additive disturbance in a system can be estimated and cancelled out if all the states are available. In practice, however, it is sometimes impossible to obtain some of the states. Thus, it is desirable that disturbance estimation can be achieved using only output measurement. This problem is considered in this section. The outline of this section is given as follows. Firstly, it is shown that, using only output measurement, estimation of disturbance is possible. Next, it is shown that, even if there is noise in the output measurement of a system, the system is robust using the disturbance estimation/cancellation method in an appropriate sense. In addition, simulation examples are given to demonstrate the methods proposed for both problems.

3.5.1 Preliminaries

In this subsection, some assumptions, which are required for the methods described in this section, are stated. In addition to Assumptions 1-5, given at Section 2.3.2 in Chapter 2, the following assumption is required.

Assumption 7 *(A, C) is an observable pair.*

3.5.2 Estimation of disturbance by output measurement

In this subsection, using only output measurement, estimation of the disturbance can be achieved. Firstly, the problem formulation is given. In order to estimate the disturbance, two problems are required to be overcome. It is shown that an adaptive algorithm can be defined in terms of only output measurement. Next, it is shown that the estimated disturbance can be reconstructed using control inputs to the observer-like system. Finally, a simulation example is shown to demonstrate the arguments in this section.

Recall that the system, feedforward filter and modified reference signal are defined as follows (see (2.2.5), (2.2.8), and (2.2.9)). Here, the system is modelled by

$$\dot{r}(t) = Ar(t) + B(p(t) + u(t)), \tag{3.5.1}$$

$$y_r(t) = Cr(t). \tag{3.5.2}$$

The feedforward filter is given by

$$\begin{aligned} \dot{x}_f(t) &= Ax_f(t) + Bu(t), \\ y_{xf}(t) &= Cx_f(t). \end{aligned}$$

The modified reference signal is generated by

$$\bar{r}(t) = r(t) - x_f(t),$$

and so

$$\begin{aligned} \dot{\bar{r}}(t) &= A\bar{r}(t) + Bp(t), \\ y_{\bar{r}}(t) &= C\bar{r}(t). \end{aligned}$$

At this time, since the states of the system are not available, the observer-like system is defined as follows:

$$\begin{aligned} \dot{\bar{x}}(t) &= A\bar{x}(t) + \bar{u}(t), \\ y_{\bar{x}}(t) &= C\bar{x}(t), \end{aligned} \tag{3.5.3}$$

where $\bar{u}(t) = G(t)e_o$, $G(t)$ is to be determined, and

$$e_o(t) = y_{\bar{x}}(t) - y_r(t).$$

The error system is defined by

$$e(t) = x(t) - \bar{x}(t)$$

and

$$\dot{e}(t) = (A + G(t)C)e(t) - Bp(t). \tag{3.5.4}$$

Hence, in terms of the error system, if the matrix $G(t)$ can be determined using an adaptive law, estimation of the disturbance is possible. However, the following questions to be addressed:

1. How can one construct the adaptive algorithm? Note that the states are not available and, therefore, the signal $||\bar{e}(t)||$ is not available.
2. How can one determine an estimate of $p(t)$? Note that, since the input matrix of the observer-like system and system (3.5.1) are different, the input to the observer-like system is not the estimate of $p(t)$.

These questions can be answered. For the first question, in terms of construction of the adaptive algorithm, $||e_o(t)|| = ||y_{\bar{x}}(t) - y_r(t)||$ can be used instead of $||\bar{e}(t)||$ in Algorithm 3 as a consequence of the robust property of the system. For the second question, relating to the difference of the input matrices of the system and the observer-like system, this problem can be solved using the method for estimation and cancellation of residual uncertainty. Thus, estimation of $p(t)$ can be done. Details are shown later.

Adaptive algorithm

Algorithm 5 *i) One of the eigenvalue of the error system, say $\lambda_1(t)$, is determined as follows. Suppose δ and ϵ_e are specified constants, which are determined by control designer. Define $V(t) := ||e_o(t)||^2$. At $t = t_0$, $\lambda_1(t) := -\lambda_0$ and $\dot{\lambda}_1(t) := -\lambda_{d0}$, where $\lambda_0 \in \mathbb{R}^+$ and $\lambda_{d0} := 0$ or δ are chosen by the control designer. The structure of $\dot{\lambda}_1(t)$ is determined as follows:*

1. *let $\tau := t$ $(t \geq t_0)$;*
2. *evaluate $\bar{e}(\tau)$ and, hence, $V(\tau)$ is also obtained;*
3. *(a) if $(V(\tau) \leq \epsilon_e^2$ and $\dot{\lambda}_1(\tau) = -\delta)$ then $\tau_1 = \tau$ and $\dot{\lambda}_1(\cdot)$ has the following structure: $\dot{\lambda}_1(s) = f(s, \tau_1)$ for $s \geq \tau_1$;*
   (b) or if $(V(\tau) > \epsilon_e^2$ and $\dot{\lambda}_1(\tau) = 0)$ then $\tau_1 = \tau$ and $\dot{\lambda}_1(\cdot)$ has the structure: $\dot{\lambda}_1(s) = g(s, \tau_1)$ for $s \geq \tau_1$;
   (c) otherwise, the structure of $\dot{\lambda}_1(\cdot)$ is not changed;

4. $t = t + \Delta t$ where Δt is a prescribed positive constant;

5. evaluate $\dot{\lambda}_1(t)$ using the given structure of $\dot{\lambda}_1(s)$;

where $f(t,\tau)$ and $g(t,\tau)$ are defined by

$$f(t,\tau) := \begin{cases} \frac{1}{2}\delta \sin(\omega(t-\tau) + \frac{3}{2}\pi) - \frac{1}{2}\delta, & \tau \le t \le \frac{\pi}{\omega} + \tau, \\ 0, & \frac{\pi}{\omega} + \tau < t, \end{cases}$$

$$g(t,\tau) := \begin{cases} \frac{1}{2}\delta \sin(\omega(t-\tau) + \frac{1}{2}\pi) - \frac{1}{2}\delta, & \tau \le t \le \frac{\pi}{\omega} + \tau, \\ -\delta, & \frac{\pi}{\omega} + \tau < t. \end{cases}$$

ii) The remaining eigenvalues, say $\lambda_2(t) \cdots \lambda_n(t)$, are determined by the ratios

$$\lambda_1(t) : \lambda_2(t) : \cdots : \lambda_n(t) = 1 : \kappa_2 : \cdots : \kappa_n,$$

where κ_i $(i = 2 \cdots n)$ are prescribed positive constants determined by control designer and $\kappa_i \neq 1$ for all i with $\kappa_i \neq \kappa_j$ for $i \neq j$.

iii) In addition to the above criteria, at the initial time $t = t_0$, all the eigenvalues are assigned so that $\lambda_i(t) < -1$ for $i \ge 1$ and $t \ge t_0$.

Remark 32 Using the eigenvalues determined by Algorithm 5, the feedback gain of the observer-like system is by

$$G(t)C = (T(t)\Lambda(t)T^{-1}(t))^t - A \tag{3.5.5}$$

where A is the system matrix of the system in observable canonical form, $\Lambda(t) = diag(\lambda_1(t), \cdots, \lambda_n(t))$, and $T(t)$ is determined by (2.5.4) or (2.6.5).

Remark 33 In Chapter 2, the analysis is performed with respect to a system matrix in controllable canonical form. However, in this chapter, the system matrix should be in an observable canonical form, since the feedback to an observer-like system is in terms of output. Thus, the matrix used to diagonalise the system matrix is different from the matrices in Chapter 2. However, the analysis of the previous chapter can still be applied. Consider the following equation:

$$V^{-1}(t)A_o(t)V(t) = T^{-1}(t)A_c(t)T(t) = \Lambda(t), \tag{3.5.6}$$

where $A_o(t)$ is the system matrix which is expressed in observable canonical form, $A_c(t)$ is a system matrix which is expressed in controllable canonical form, and $\Lambda(t)$ is the diagonalised system matrix. Since $A_o^t(t) = A_c(t)$, (3.5.6) is expressed as follows:

$$\begin{aligned} \Lambda(t) &= V^{-1}(t)A_c^t(t)V(t) \\ &= \left(T^{-1}(t)A_c(t)T(t)\right)^t \\ &= T^t(t)A_c^t(t)(T^t(t))^{-1}. \end{aligned}$$

Thus,

$$V(t) = (T^t(t))^{-1}$$

and

$$V^{-1}(t) = T^t(t).$$

Using these relations, it is shown that $||V^{-1}(t)B_o||_i$ and $||V^{-1}(t)\dot{V}(t)||_1$ ($i =$ 1 *or* 2) have the same characteristics as those for the matrices in controllable canonical form.

By definition,

$$V^{-1}(t)B_o = T^{-1}(t)B_c,$$

where B_o is the input matrix in observable canonical form and B_c is the input matrix in controllable canonical form. It follows that

$$||V^{-1}(t)B_o||_i = ||T^{-1}(t)B_c||_i$$

with $i = 1$ *or* 2. Therefore, $||V^{-1}(t)B_o||_i$ has the same characteristics as $||T^{-1}(t)B_c||_i$.

By definition,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[T^{-1}(t)T(t)\right] = \dot{T}^{-1}(t)T(t) + T^{-1}(t)\dot{T}(t) = 0$$

and so

$$\dot{T}^{-1}(t)T(t) = -T^{-1}(t)\dot{T}(t). \tag{3.5.7}$$

It has already been shown that

$$V^{-1}(t)\dot{V}(t) = T^t(t)(\dot{T}^{-1}(t))^t \tag{3.5.8}$$

and so (3.5.7) and (3.5.8) imply that

$$\begin{aligned} V^{-1}(t)\dot{V}(t) &= (\dot{T}^{-1}(t)T(t))^t \\ &= -(T^{-1}(t)\dot{T}(t))^t. \end{aligned}$$

It follows that

$$||V^{-1}(t)\dot{V}(t)||_1 = ||T^{-1}(t)\dot{T}^t(t)||_1.$$

Therefore, $||V^{-1}(t)\dot{V}(t)||_1$ has the same characteristics as $||T^{-1}(t)\dot{T}^t(t)||_1$. Hence, the analysis of Chapter 2 can be applied to the problem of estimation by output.

Remark 34 All the analysis in Section 3.2 can be applied to estimation of the disturbance by output because of the following reasons. Since $e_o(t) = CT(t)\bar{e}(t)$, all the analysis corresponding to the inequality $||e(t)|| \leq ||T(t)||_1\, ||\bar{e}(t)||$, in Section 3.2, can be applied using $||e_o(t)|| \leq ||C||\, ||T(t)||_1\, ||\bar{e}(t)||$ instead of using the former inequality. Moreover, in view of Remark 33, the analysis corresponding to $T^{-1}(t)B$ and $T^{-1}(t)\dot{T}(t)$ can also be applied for estimation of the disturbance by output. Therefore, all the analysis in Section 3.2 can be applied to the problem of estimation of disturbance by output.

Reconstruction of disturbance

Reconstruction of the estimated disturbance can be performed in the same way as for cancellation of residual disturbance. Recall the system, the observer-like system, and the error system:

$$
\begin{aligned}
\dot{r}(t) &= Ar(t) + B(u(t) + p(t)), \\
y_r(t) &= Cr(t), \\
\dot{x}(t) &= Ax(t) + G(t)Ce(t) \\
&= Ax(t) + I\bar{u}(t), \\
y_x(t) &= Cx(t), \\
\dot{e}(t) &= Ae(t) + I\bar{u}(t) - Bp(t), \\
y_e(t) &= Ce(t),
\end{aligned}
$$

where $r(t)$ is the state of the system, $y_r(t)$ is the output of the system, $x(t)$ is the state of the observer-like system, $y_x(t)$ is the output of the observer-like system, $e(t)$ is the state of the error system, and $y_e(t)$ is the output of error system. The input-output relations of these systems are given as follows:

$$
\begin{aligned}
Y_r(s) &= C(sI - A)^{-1}B(P(s) + U(s)) \\
&= G_1(s)(P(s) + U(s)) \\
Y_x(s) &= C(sI - A)^{-1}I\bar{U}(s) \\
&= G_2\bar{U}(s)
\end{aligned}
\tag{3.5.9}
$$

$$
\begin{aligned}
Y_e(s) &= C(sI - A)^{-1}I\bar{U}(s) - C(sI - A)^{-1}P(s) \\
&= -G_1(s)P(s) + G_2(s)\bar{U}(s)
\end{aligned}
\tag{3.5.10}
$$

where $G_i(s)$ $(i = 1, 2)$ are transfer function matrices of appropriate dimensions, $Y_r(s) = \mathcal{L}\{y_r(t)\}$, $Y_x(s) = \mathcal{L}\{y_x(t)\}$, $Y_e(s) = \mathcal{L}\{y_e(t)\}$, $P(s) = \mathcal{L}\{p(t)\}$, $U(s) = \mathcal{L}\{u(t)\}$, and $\bar{U}(s) = \mathcal{L}\{\bar{u}(t)\}$.

For t sufficiently large, as a result of Algorithm 5 and in view of the comments in Remark 34, the following relation holds:

$$||y_e(t)|| \approx 0.$$

Define $U(s)$ as follows:

$$U(s) := -G_1^{-1}(s)G_2(s)\bar{U}(s). \tag{3.5.11}$$

Substituting (3.5.11) into (3.5.9),

$$Y_r(s) = G_1(s)P(s) - G_2(s)\bar{U}(s). \tag{3.5.12}$$

By (3.5.10) and (3.5.12), the input-output relation between $P(s)$ and $\bar{U}(s)$ to $Y_r(s)$ is the same as the input-output relation from $P(s)$ and $\bar{U}(s)$ to $Y_e(s)$, except that there is a difference of sign. Thus, for t sufficiently large,

$$||y_r(t)|| \approx 0.$$

Thus, in view of (3.5.9), $-u(t) \approx p(t)$ for t sufficiently large.

It has already been shown that the adaptive algorithm can be applied using only output measurement. Moreover, the disturbance is able to be estimated using control inputs to the observer-like system. Therefore, disturbance estimation/cancellation can be achieved using only output measurement.

Simulation example

The system, to be examined, is a second order single-input/single-output linear system expressed as follows:

$$\dot{r}(t) = Ar(t) + B(u(t) + d(t)),$$
$$y_r(t) = Cr(t),$$

where $d(t)$ expresses the external disturbance/uncertainty, and the system matrices are given by

$$A = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [0 \quad 1].$$

For simulation purposes, an initial condition for the system is taken to be $r(t_0) = [0\ 0]^t$. The objective of control is to design a controller in order to track the reference signal $R(t) = 1$ by the output of the system, namely $y_r(t) = r_2(t)$. The control input for this tracking, which is represented as $u_{tr}(t)$, is designed as follows. The transfer function of the system is given as follows:

$$G(s) = C(sI - A)^{-1}B = \frac{1}{(s+2)(s+3)}$$

Hence, the gain of this transfer function with respect to constant signal is given by

$$|G(0)| = \frac{1}{6}.$$

Thus, the control input for the tracking is given as follows:

$$u_{tr}(t) = 1.0 \times 6.0.$$

For this problem, an accuracy specified by $\epsilon_e^2 = 2.0 \times 10^{-5}$ is required. For the adaptive algorithm, $\delta = -2.0$, $\kappa_2 = 1.5$, and $\omega = 10$ are used and initially, $\dot{\lambda}_1(t_0) = 0$ and $\lambda_1(t_0) = -2.0$ are set. For simulation purposes, the disturbance term is chosen to be $d(t) = 5 + 2r_1(t) - 6r_2(t)$, where $r_i(\cdot)$ are the components of $r(t) = [r_1(t)\ r_2(t)]^t$. Also, the estimated disturbance is used to cancel out the effect of disturbance to the system; i.e. the opposite sign of the estimated disturbance is fed back to the system in order to cancel out the effect of the disturbance. The simulation has been performed with the following configuration:

Programing language: C++;

Compiler: g++ version 3.2.2;

Algorithm to obtain numerical solution of ODE: Runge-Kutta (see [31]);

Time step for the Runge-Kutta algorithm: 1.0×10^{-4}.

The open-loop response for the output of the system and the reference signal are shown in Figure 3.5.1. The solid line represents the output of the system and dashed line represents the reference signal for tracking. From this figure, it is clear that the tracking error is too large and, hence, in the presence of uncertainty and disturbance, tracking has not been achieved

The actual and estimated disturbances are shown in Figure 3.5.4. The solid line represents the actual disturbance and the dashed line represents the estimated disturbance. From this figure, it is observed that the estimated disturbance converges to the actual one very rapidly. The closed-loop response of the output of the system and reference signal are shown in Figure 3.5.2. The solid line represents the output of the system and dashed line represents the reference signal. In this figure, it is observed that the output converges to the reference signal despite the existence of uncertainty and disturbance. The output and reference signal at later time are shown in Figure 3.5.3. The solid line represents the output of the system and the dashed line represents the reference signal. From this figure, one can conclude that although there is some tracking error, the error is small. Thus, estimating and cancelling the disturbance has been achieved and its performance is good.

The histories of the eigenvalues of the error system (3.5.4) and the gains of the observer-like system (3.5.3) are shown in Figure 3.5.5 and 3.5.6. In these figures, it is observed that their values are decreased until they reach certain values and then they remain there. The behaviour of the value of the Lyapunov-like function is shown in Figure 3.5.7. In this figure, it is observed that the value is decreased very rapidly. Behaviour of Lyapunov-like function at later time is shown at Figure 3.5.8. In this figure, it is observed that the value of the function remains less than certain value; i.e. $V \leq \epsilon_c^2 = 2 \times 10^{-8}$. Thus, one can conclude that the eigenvalues of the error system, the gains of the observer-like system, and the value of the Lyapunov-like function have the properties described in Algorithm 5, Lemma 24, and Remark 34.

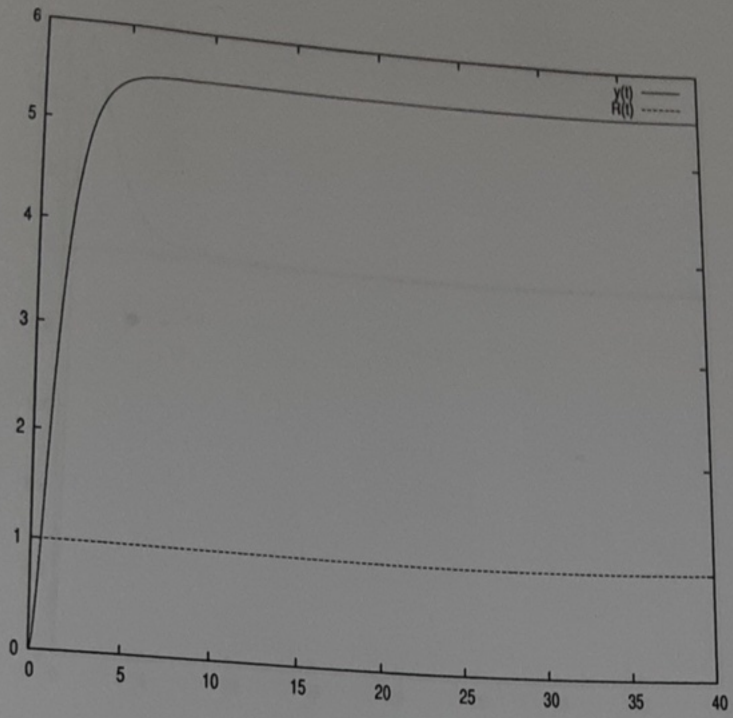

Figure 3.5.1: Open-loop response of the output $y_r(t)$ and the reference signal $R(t)$.

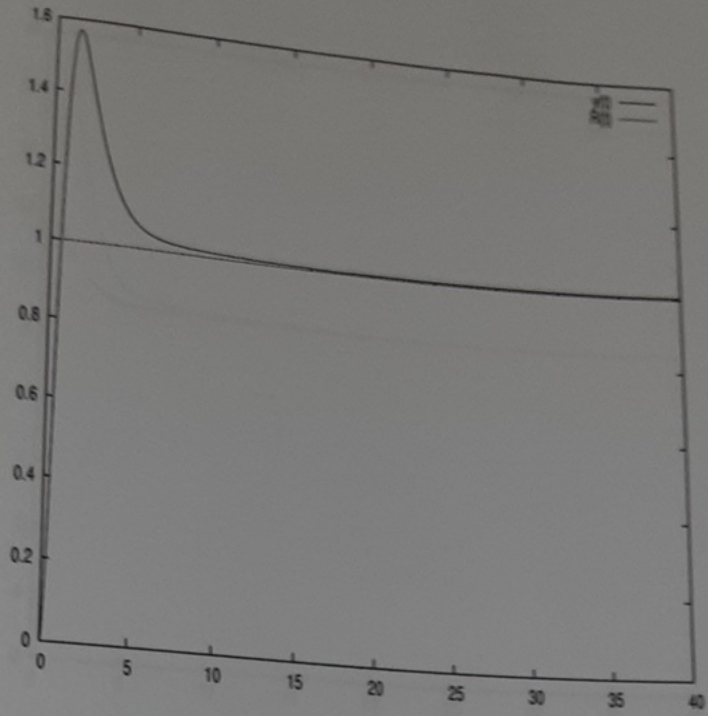

Figure 3.5.2: Closed-loop response of the state $y_r(t)$ and the reference signal $R(t)$.

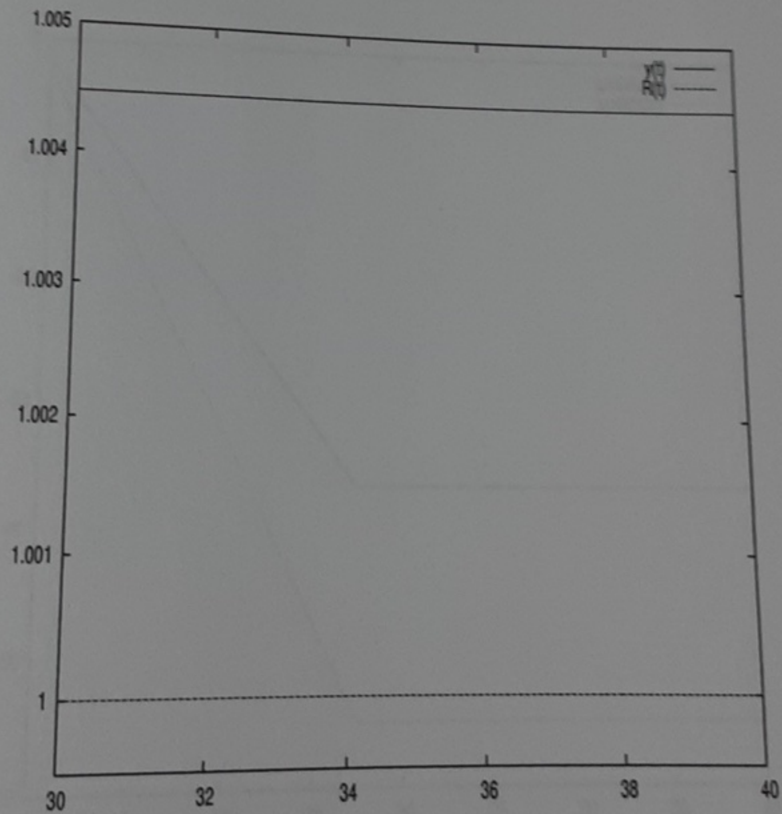

Figure 3.5.3: Closed-loop response of the state $y_r(t)$ and the reference signal $R(t)$ at some later time interval.

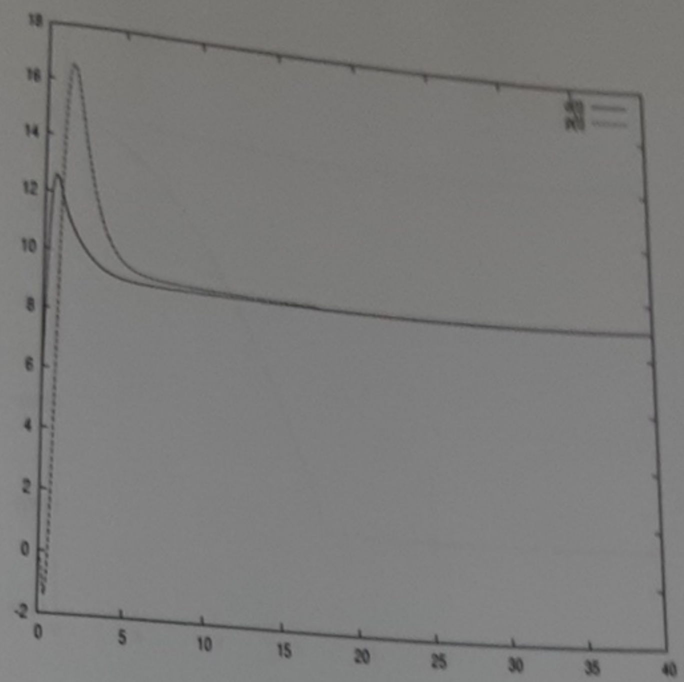

Figure 3.5.4: The actual and estimated disturbances, $d(t)$ and $p(t)$, respectively.

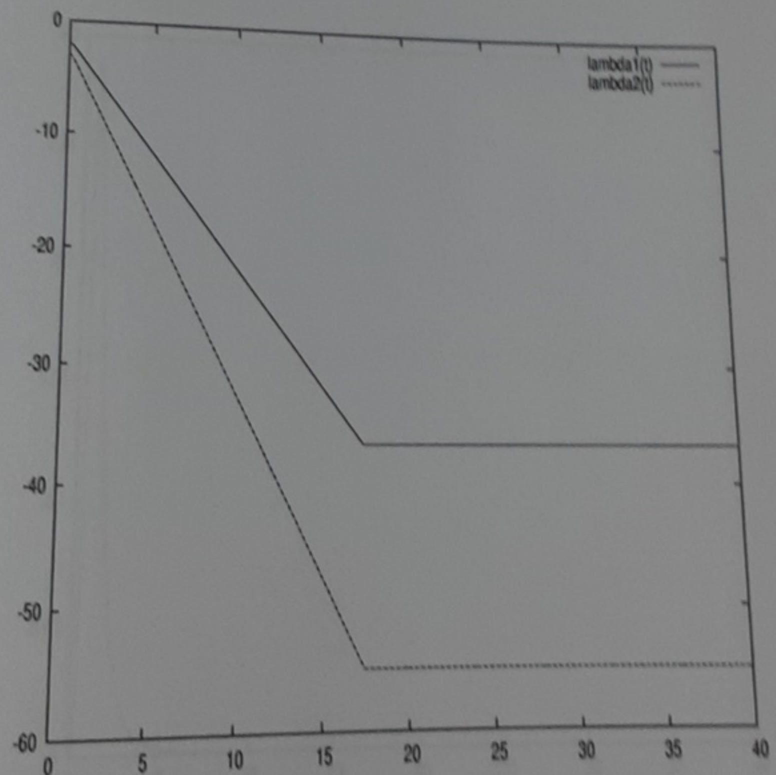

Figure 3.5.5: Histories of the eigenvalues of the error system: $\lambda_1(t)$ and $\lambda_2(t)$.

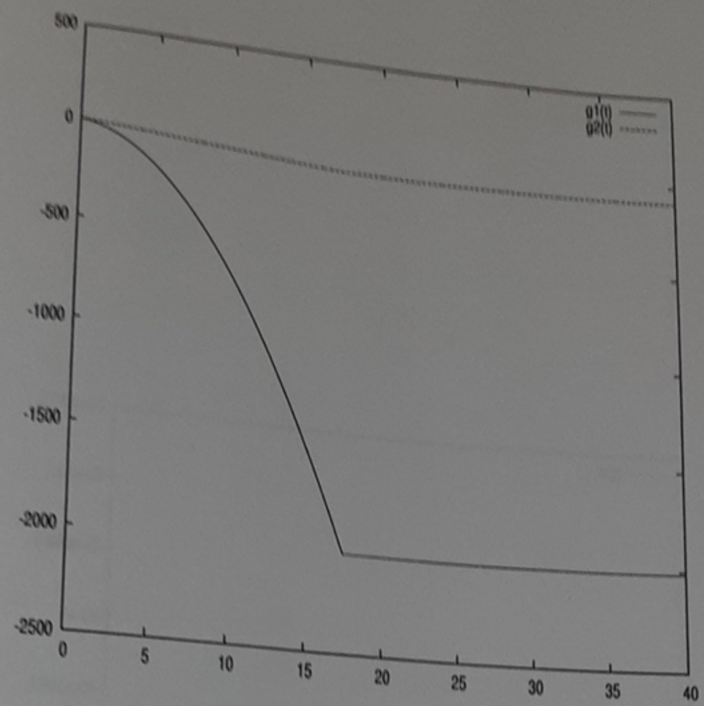

Figure 3.5.6: Histories of the feedback gains of the observer-like system, $g_1(t)$ and $g_2(t)$, respectively.

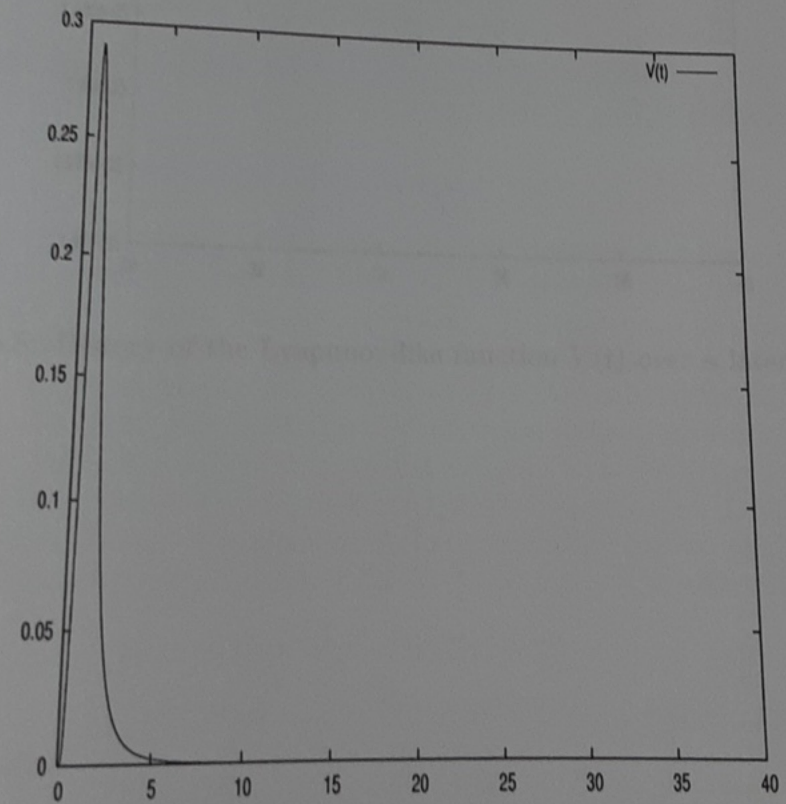

Figure 3.5.7: History of the Lyapunov-like function $V(t)$.

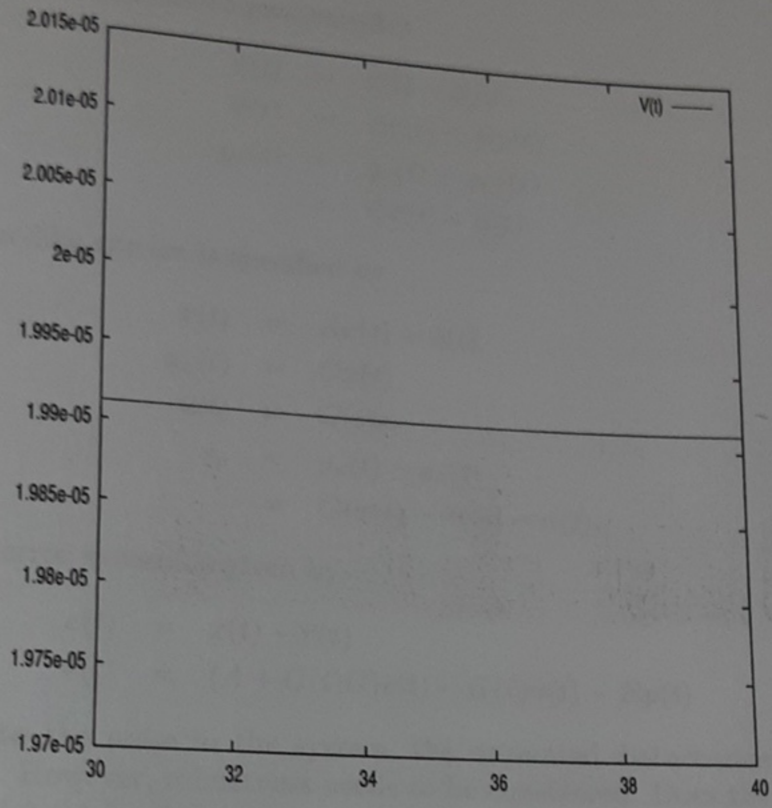

Figure 3.5.8: History of the Lyapunov-like function $V(t)$ over a later time interval.

3.5.3 Estimation with measurement noise

Consider the system described in (3.5.1) and (3.5.2) in the presence of additive noise on the output measurement which is modelled by

$$\dot{r}(t) = Ar(t) + B(p(t) + u(t)) \quad (3.5.13)$$
$$y_r(t) = Cr(t) + n(t), \quad (3.5.14)$$

where $n(t)$ is measurement noise. The feedforward filter is given by

$$\dot{x}_f(t) = Ax_f(t) + Bu(t),$$
$$y_{xf}(t) = Cx_f(t).$$

and the modified reference signal satisfies

$$\begin{aligned} \bar{r}(t) &= r(t) - x_f(t) \\ \dot{\bar{r}}(t) &= A\bar{r}(t) + Bp(t) \quad (3.5.15) \\ y_{\bar{r}}(t) &= y_r(t) - y_{xf}(t) \\ &= C\bar{r}(t) + n(t). \quad (3.5.16) \end{aligned}$$

The observer-like system is specified by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \bar{u}(t) \\ y_x(t) &= Cx(t) \\ \bar{u}(t) &= G(t)e_o \\ e_o &= y_x(t) - y_{\bar{r}}(t) \\ &= C(x(t) - \bar{r}(t)) - n(t). \end{aligned}$$

Finally, the error system is given by

$$\begin{aligned} e(t) &= x(t) - \bar{r}(t) \\ \dot{e}(t) &= (A + G(t)C)e(t) - G(t)n(t) - Bp(t) \end{aligned}$$

Hence, due to the noise in the system, the estimated disturbance is not the same as $p(t)$. However, robustness needs to be considered. Does the robustness property still hold in the presence in noise? The answer to this question is discussed next.

Loosely speaking, the actual disturbance $p(t)$ plus an additive disturbance, which has the effect of introducing $n(t)$ into the output of system (3.5.13)-(3.5.14), i.e. $y_r(t)$, will be estimated instead of $p(t)$ only. Therefore, the actual output of system (3.5.14) will be perturbed by the 'sensor' noise. A more precise analysis is given below. The input output relation of the system relating to the modified reference signal, namely (3.5.15) and (3.5.16), is represented by

$$Y_{\bar{r}}(s) = C(sI - A)^{-1}BP(s) + N(s), \quad (3.5.17)$$

where $Y_{\bar{r}}(s) = \mathcal{L}\{y_{\bar{r}}(t)\}$, $P(s) = \mathcal{L}\{p(t)\}$, and $N(s) = \mathcal{L}\{n(t)\}$. Consider the following system:

$$\begin{aligned} \dot{\bar{r}}_2(t) &= A\bar{r}_2(t) + B(p(t) + p_n(t)), \\ y_{\bar{r}_2}(t) &= C\bar{r}_2(t), \end{aligned} \quad (3.5.18)$$

where $p_n(t)$ is defined later. Suppose the following relation holds for $y_{r_2}(t)$:

$$y_{r_2}(t) = y_r(t), \quad \forall t \geq t_0.$$

The input-output relation of this system is

$$\mathcal{L}\{Y_{r_2}(s)\} = C(sI-A)^{-1}BP(s) + C(sI-A)^{-1}BP_n(s).$$

Since by definition, $y_{r_2}(t) = y_r(t)$ for all $t \geq t_0$, it follows from (3.5.17) that

$$C(sI-A)^{-1}BP_n(s) = N(s).$$

Thus, $p_n(t)$ produces $n(t)$ to the output (3.5.18). If the output from the observer-like system satisfies $y_x(t) \approx y_r(t)$, then $y_x(t) \approx y_{r_2}(t)$. Thus, the disturbance estimation method estimates $p(t) + p_n(t)$.

Define the estimate of $p(t) + p_n(t)$ to be $\bar{p}(t) + \bar{p}_n(t)$. When the estimated disturbance is fed back to system (3.5.13), the following equations are obtained:

$$\begin{aligned}
\dot{r}(t) &= Ar(t) + B(p(t) + u(t)) \\
&= Ar(t) + B(p(t) - \bar{p}(t) - \bar{p}_n(t)) \\
&= Ar(t) + B(\epsilon(t) - \bar{p}_n(t)) \\
y_r(t) &= Cr(t) + n(t)
\end{aligned}$$

where the signal $\epsilon(t)$ is 'small' values. The input-output relation of this system is

$$\begin{aligned}
Y_r(s) &= C(sI-A)^{-1}BE(s) - C(sI-A)^{-1}B\bar{P}_n(s) + N(s) \\
&= C(sI-A)^{-1}BE(s) - \bar{N}(s) + N(s)
\end{aligned}$$

where $E(s) = \mathcal{L}\{\epsilon(t)\}$, $\bar{P}_n(s) = \mathcal{L}\{\bar{p}_n(t)\}$, $\bar{N}(s) = C(sI-A)^{-1}B\bar{P}_n(s)$, and $N(s) = \mathcal{L}\{n(t)\}$. Note that $\bar{n}(t)$ tends to $n(t)$.

This equation indicates that, in the presence of noise and when the disturbance estimation method is applied,

1. the effect of the disturbance is cancelled out;
2. the actual output of system (3.5.14) is perturbed by noise which is close to the measurement noise.

These two statements imply that, in the presence of the noise and disturbance and when the disturbance cancellation method is applied, the output of system (3.5.14) is perturbed by a signal which is almost the same signal as the measurement noise. In practice, the amplitude of the measurement noise, whose upper bound is not known, may not be large. Therefore, it is concluded that using the disturbance estimation method, system (3.5.13)-(3.5.14) is robust in the presence of measurement noise.

Simulation example

The system, to be examined, is a second order single-input/single-output linear system expressed by

$$\dot{r}(t) = Ar(t) + B(u(t) + d(t)), \tag{3.5.19}$$

$$y_r(t) = Cr(t) + n(t), \tag{3.5.20}$$

where $d(t)$ is the external disturbance/uncertainty and $n(t)$ is the measurement noise. The system matrices are given by

$$A = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

For simulation purposes, an initial condition for the system is taken to be $r(t_0) = [0\ 0]^t$. The objective of the control is to track the reference signal $R(t) = 1$ by the output of the system, $y_r(t) = r_2(t)$. The control input for the tracking, which is represented by $u_{tr}(t)$, is determined by the following procedure. The transfer function between $u_{tr}(t)$ to $y_r(t)$ of system (3.5.19)-(3.5.20) is given by

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \frac{1}{(s+2)(s+3)}. \end{aligned}$$

Hence, steady state gain of this transfer function with respect to a constant signal is given by

$$|G(0)| = \frac{1}{6}.$$

Thus, to track the reference signal, $R(t) = 1$, the tracking control input is determined as follows:

$$u_{tr}(t) = 1.0 \times 6.0.$$

For this problem, an accuracy specified by $\epsilon_e^2 = 2.0 \times 10^{-5}$ is required. For the adaptive algorithm, $\delta = -2.0$, $\kappa_2 = 1.5$, and $\omega = 10$ are used and, initially, the eigenvalues are chosen to satisfy $\dot{\lambda}_1(t_0) = 0$ and $\lambda_1(t_0) = -2.0$. For simulation purposes, the disturbance term is chosen to be $d(t) = 5 + 2r_1(t) - 6r_2(t)$, where $r_i(\cdot)$ are the components of $r(t) = [r_1(t)\ r_2(t)]^t$. Also, the measurement noise is chosen as $n(t) = 1.0 \times 10^{-3}\sin(1000t)$. Also, the estimated disturbance is used to cancel out the effect of disturbance to the system; i.e. the opposite sign of the estimated disturbance is fed back to the system in order to cancel out the effect of the disturbance. The simulation has been performed with the following configuration:

Programing language: C++;

Compiler: g++ version 3.2.2;

Algorithm to obtain the numerical solution of the ODE: Runge-Kutta (see [31]);

Time step for the Runge-Kutta algorithm: 1.0×10^{-4}.

The open-loop response of the output of the system is shown in Figure 3.5.9. The solid line represents the output of the system and the dashed line represents the reference signal for tracking. From this figure, it is clear that, because of the presence of uncertainty and disturbance, tracking has failed for the open-loop system.

Estimation error of disturbance is shown in Figure 3.5.12. In this figure, it is observed that, unlike the noise-free system, the difference between the actual and estimated disturbances are large. This is because the estimated disturbances contains an additional disturbance which produces noise to the output as described in the previous subsection. The output of the system and reference signal for tracking are shown in Figure 3.5.10. The solid line represents the output of the system and the dashed line represents the reference signal. In this figure, it is observed that the output of the system converges to the reference signal rapidly. The output of the system and reference signal over some later time are shown in Figure 3.5.11. The solid line represents the output of the system and the dashed line represents the reference signal. In this figure, two things are observed. Firstly, although there is some tracking error, the error is 'small'. Secondly, the output of the system is perturbed by measurement noise but its amplitude of perturbation is smaller than the amplitude of the actual measurement noise. Therefore, estimation and cancellation of disturbance has succeeded.

The histories of the eigenvalues of the error system and the gains of the observer-like system are shown in Figure 3.5.13 and 3.5.14. In these figures, it is observed that their values are decreased until they reach certain values and then they remain at those values. The history of the Lyapunov-like function is shown in Figure 3.5.15 and 3.5.16. In Figure 3.5.15, it is observed that the value of this function decreases very rapidly. The history of value of this function over some later time interval is shown in Figure 3.5.16. In this figure, it is observed that the value of $V(t)$ remain within a certain range, that is $V \leq \epsilon_e^2 = 2.0^{-5}$. Thus, the graphs illustrate that the eigenvalues of the error system, the gains of the observer-like system, and value of Lyapunov-like function have the properties described by Algorithm 5, Lemma 24, and Remark 34.

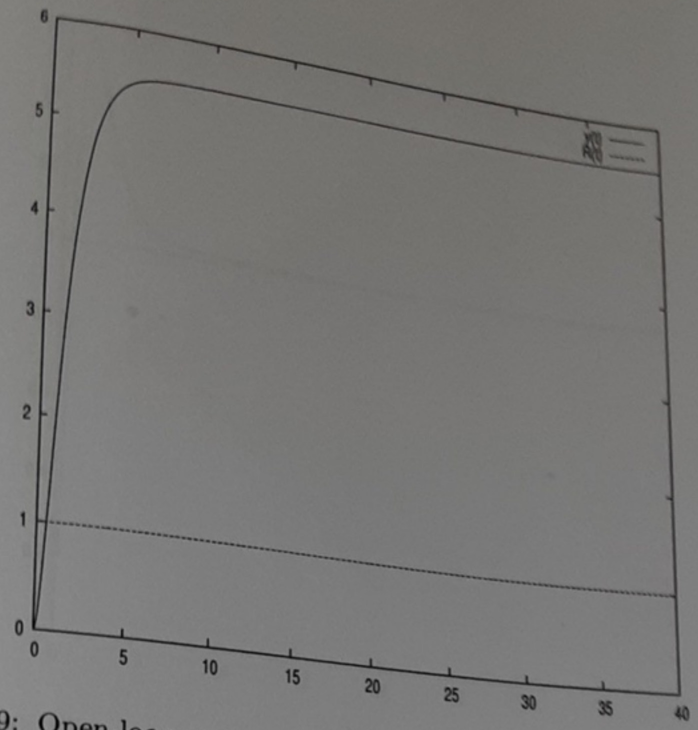

Figure 3.5.9: Open-loop response of the output $y_r(t)$ and the reference signal $R(t)$.

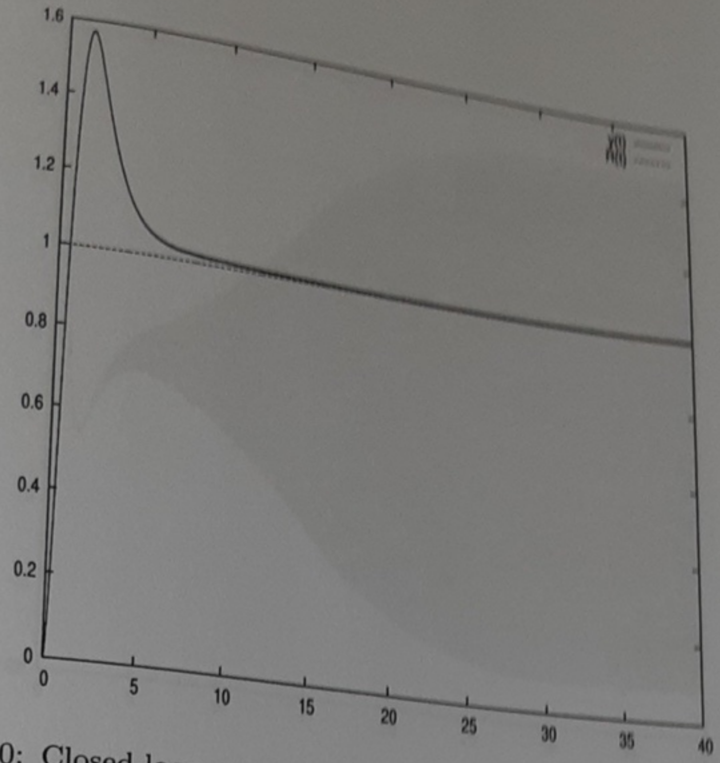

Figure 3.5.10: Closed-loop response of the state $y_r(t)$ and the reference signal $R(t)$.

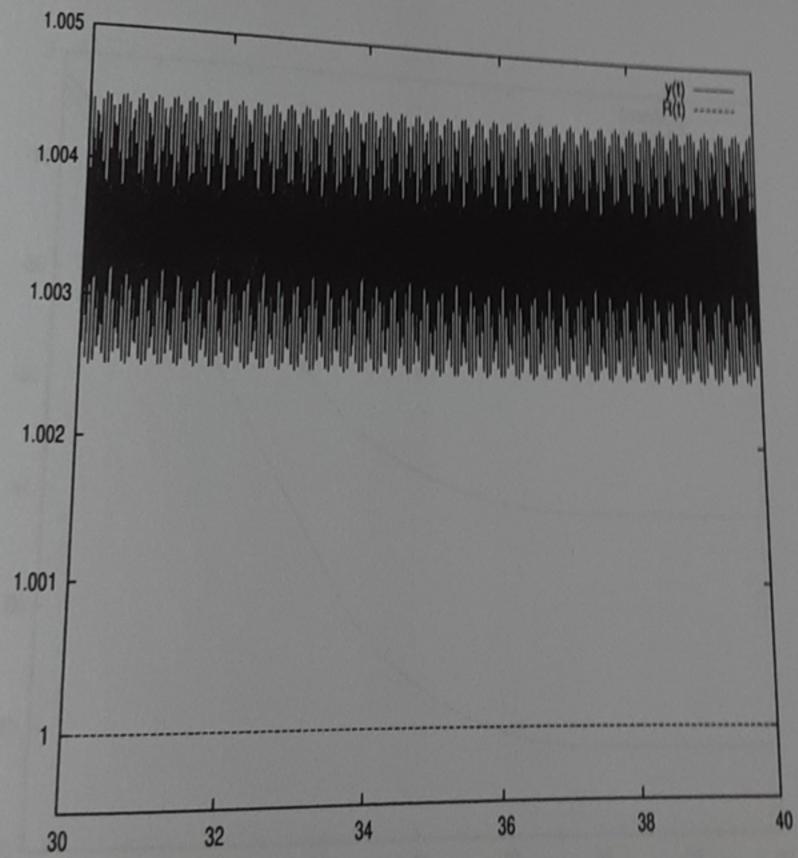

Figure 3.5.11: Closed-loop response of the state $y_r(t)$ and the reference signal $R(t)$ over some later time period.

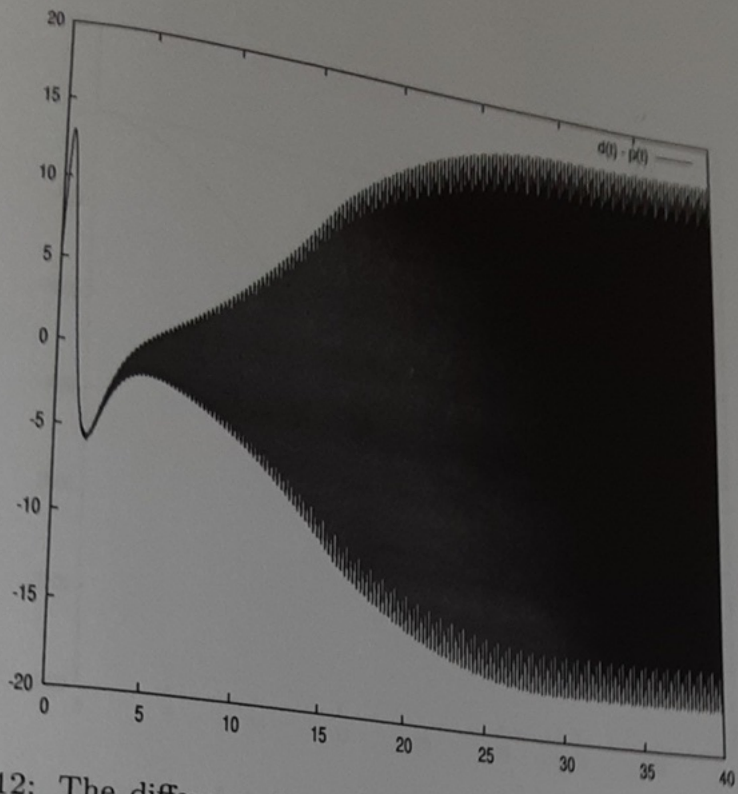

Figure 3.5.12: The difference between the actual and estimated disturbance, $d(t) - p(t)$.

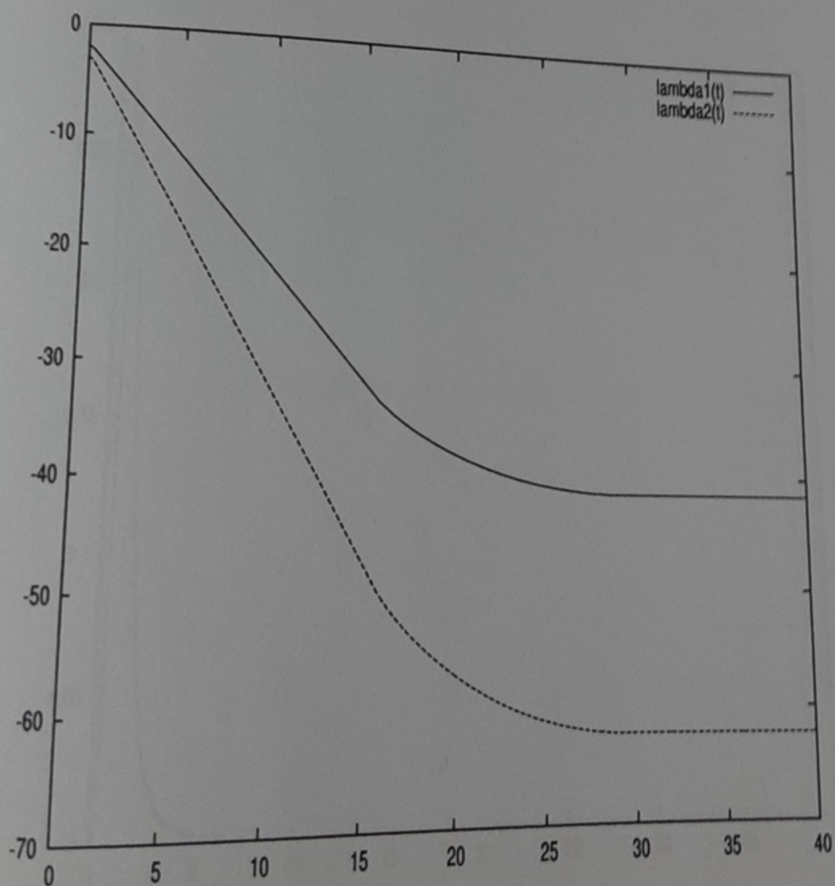

Figure 3.5.13: Histories of the eigenvalues of the error system: $\lambda_1(t)$ and $\lambda_2(t)$.

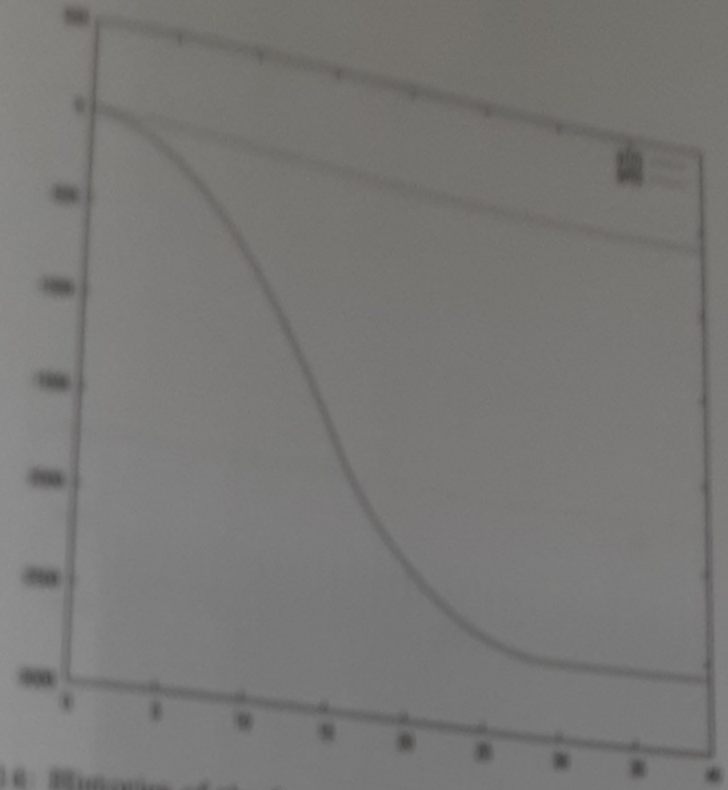

Figure 3.5.14: Histories of the feedback gains of the observer-like system: $g_1(t)$ and $g_2(t)$.

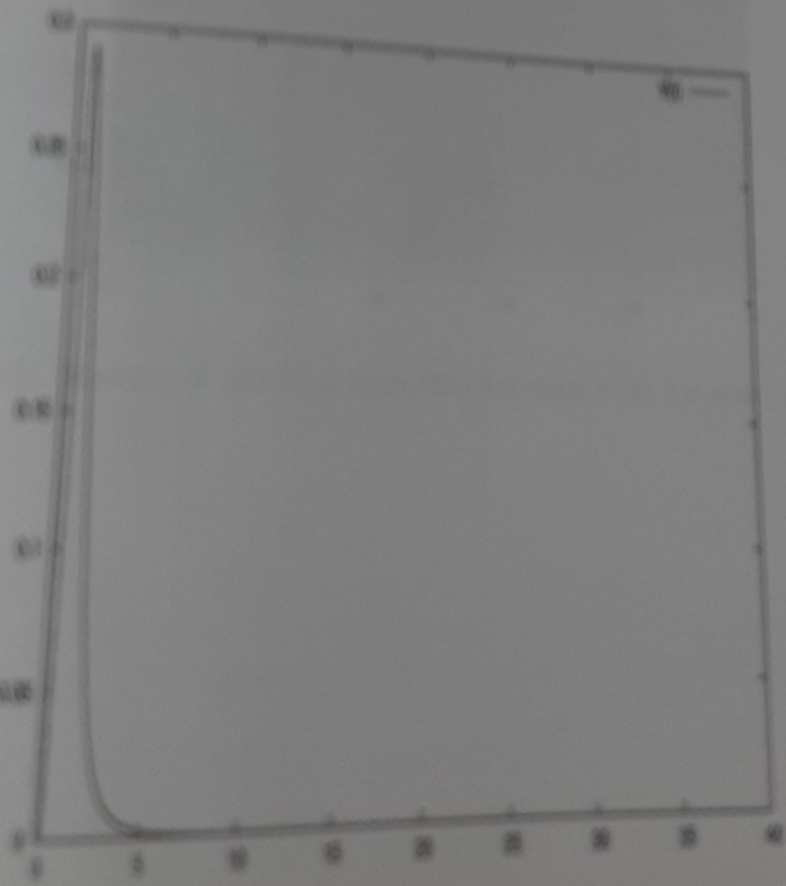

Figure 3.5.15: History of the Lyapunov-like function $V(t)$.

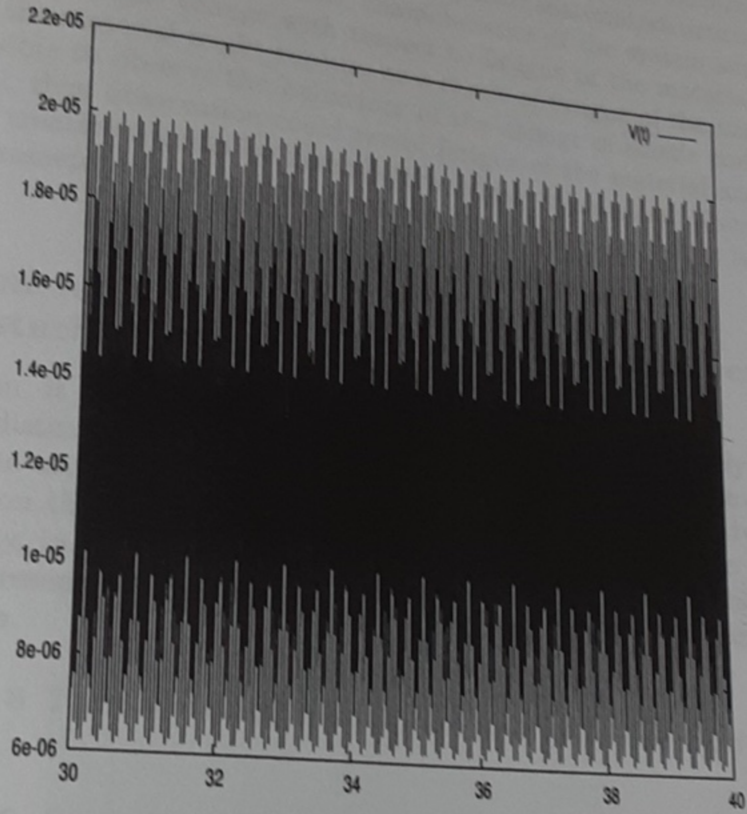

Figure 3.5.16: History of the Lyapunov-like function $V(t)$ for some later time interval.

3.6 Estimation of the parameters generating parametric uncertainty from the estimated disturbance for single-input systems

In this section, a method to estimate the parameter variation caused by parametric uncertainty from the estimated disturbance is introduced. If an objective of the control designer is to design for a nominal system, it is enough to estimate and cancel out disturbance/uncertainty. However, estimating parameter variation from the estimated disturbance can have a significant advantage for a specific class of systems. For instance, when a smart material/structural system is controlled in a practical situation, characteristics of the system such as the elastic constant, might change with respect to fatigue of the material. In the worst case, the material might rupture as a result of fatigue of the material. If it were possible to observe the behaviour of the change of elastic constant for the material, then observation could reveal fatigue of the material and, hence, appropriate treatment could be taken before fatigue takes place. Therefore, estimating parameter variation from the estimated disturbance might have some benefit.

3.6.1 Constant parametric uncertainty without external disturbance

In this section, it is shown that if parametric uncertainty is the only factor of uncertainty/disturbance, it is always possible to estimate the parameter variation from the estimated disturbance. Although this method is derived under the assumption that there are no external disturbances other than parametric uncertainty, in later subsections, it will be shown that this method can be used in the presence of periodic and constant disturbances under appropriate an assumption.

Assumption 8 *There are no external disturbances and unmodelled dynamics in the system.*

Assumption 9 *Parameter variation in the system is constant.*

Consider the single-input system:

$$\dot{r}(t) = Ar(t) + B\left[u(t) + d(t)\right]$$

where $r(t) \in \mathbb{R}^{2n}$, $u(t) \in \mathbb{R}$, $d(t) \in \mathbb{R}$, and A and B have appropriate dimensions. Also, A and B are expressed in controllable canonical form. Under Assumptions 8 and 9, the estimated disturbance $d(t)$ of this system is expressed as follows.

$$d(t) = a_1 r_1(t) + \cdots + a_{2n} r_{2n}(t)$$

where $r_i(t)$ $(i = 1 \cdots 2n)$ are states of the system.

Consider the situation in which the disturbance estimation/cancellation method is applied to the system and $r_1(t)$ tracks the reference signal $R(t) = \sin(\omega t)$, where ω is any positive constant. Hence, if perfect tracking is achieved then, for t sufficiently large, the states are given by

$$
\begin{aligned}
r_1(t) &= \sin(\omega t) \\
r_2(t) &= \omega\cos(\omega t) \\
&\vdots \\
r_{2n-1}(t) &= (-1)^{n+1}\omega^{2n-2}\sin(\omega t) \\
r_{2n}(t) &= (-1)^{n+1}\omega^{2n-1}\cos(\omega t)
\end{aligned}
$$

Consider integration of $d(t)$ over the period $0 \le t \le \frac{1}{\omega}\pi$. Since $\int_0^{\frac{1}{\omega}\pi}\sin(\omega t)\mathrm{d}t = \frac{2}{\omega}$ and $\int_0^{\frac{1}{\omega}\pi}\cos(\omega t)\mathrm{d}t = 0$, it follows that

$$\int_0^{\frac{1}{\omega}\pi} d(t)\mathrm{d}t = \frac{2a_1}{\omega} - 2a_3\omega + 2a_5\omega^3 + \cdots + (-1)^{n+1}2a_{2n-1}\omega^{2n-3}$$

Define

$$D_{1i} := \int_0^{\frac{1}{\omega_i}\pi} d(t)\mathrm{d}t \tag{3.6.1}$$

Note that in practice, D_{1i} are obtained numerically for a specified set of n values of ω, namely $\{\omega_1, \omega_2, \cdots, \omega_n\}$. Then D_{1i} satisfy the n equations:

$$2\begin{bmatrix} \frac{1}{\omega_1} & -\omega_1 & \cdots & (-1)^{n+1}\omega_1^{2n-3} \\ \frac{1}{\omega_2} & -\omega_2 & \cdots & (-1)^{n+1}\omega_2^{2n-3} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\omega_n} & -\omega_n & \cdots & (-1)^{n+1}\omega_n^{2n-3} \end{bmatrix}\begin{bmatrix} a_1 \\ a_3 \\ \vdots \\ a_{2n-1} \end{bmatrix} = \begin{bmatrix} D_{11} \\ D_{12} \\ \vdots \\ D_{1n} \end{bmatrix}.$$

Therefore, $[a_1 \cdots a_{2n-1}]^t$ is obtained from:

$$\begin{bmatrix} a_1 \\ a_3 \\ \vdots \\ a_{2n-1} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{\omega_1} & -\omega_1 & \cdots & (-1)^{n+1}\omega_1^{2n-3} \\ \frac{1}{\omega_2} & -\omega_2 & \cdots & (-1)^{n+1}\omega_2^{2n-3} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\omega_n} & -\omega_n & \cdots & (-1)^{n+1}\omega_n^{2n-3} \end{bmatrix}^{-1}\begin{bmatrix} D_{11} \\ D_{12} \\ \vdots \\ D_{1n} \end{bmatrix}. \tag{3.6.2}$$

Next, consider integration over the period $\frac{\pi}{2\omega} \le t \le \frac{3\pi}{2\omega}$. For this period,

$$\int_{\frac{\pi}{2\omega}}^{\frac{3\pi}{2\omega}} \sin(\omega t)\mathrm{d}t = 0$$

and

$$\int_{\frac{\pi}{2\omega}}^{\frac{3\pi}{2\omega}} \cos(\omega t)\mathrm{d}t = -\frac{2}{\omega}.$$

Therefore, integrating $d(t)$ over the period $\frac{\pi}{2\omega} \le t \le \frac{3\pi}{2\omega}$ gives

$$\int_{\frac{\pi}{2\omega}}^{\frac{3\pi}{2\omega}} d(t)\mathrm{d}t = -2a_2 + 2a_4\omega^2 + \cdots + (-1)^{n+1}2a_n\omega^{2n-2}.$$

$$D_{2i} := \int_{\frac{\pi}{2\omega_i}}^{\frac{3\pi}{2\omega_i}} d(t)\mathrm{d}t,$$

Then, the following n equations are satisfied:

$$\begin{bmatrix} D_{21} \\ D_{22} \\ \vdots \\ D_{2n} \end{bmatrix} = 2 \begin{bmatrix} -1 & \omega_1^2 & \cdots & (-1)^{n+1}\omega_1^{2n-2} \\ -1 & \omega_2^2 & \cdots & (-1)^{n+1}\omega_2^{2n-2} \\ \vdots & \vdots & \ddots & \vdots \\ -1 & \omega_n^2 & \cdots & (-1)^{n+1}\omega_n^{2n-2} \end{bmatrix} \begin{bmatrix} a_2 \\ a_4 \\ \vdots \\ a_{2n} \end{bmatrix}. \tag{3.6.3}$$

Therefore, $[a_2 \cdots a_{2n}]^t$ can be determined from

$$\begin{bmatrix} a_2 \\ a_4 \\ \vdots \\ a_{2n} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & \omega_1^2 & \cdots & (-1)^{n+1}\omega_1^{2n-2} \\ -1 & \omega_2^2 & \cdots & (-1)^{n+1}\omega_2^{2n-2} \\ \vdots & \vdots & \ddots & \vdots \\ -1 & \omega_n^2 & \cdots & (-1)^{n+1}\omega_n^{2n-2} \end{bmatrix}^{-1} \begin{bmatrix} D_{21} \\ D_{22} \\ \vdots \\ D_{2n} \end{bmatrix}. \tag{3.6.4}$$

In summary, from (3.6.2) and (3.6.4), each parameter variation, relating to the parametric uncertainty, is obtained from

$$\begin{bmatrix} a_1 \\ a_3 \\ \vdots \\ a_{2n-1} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{\omega_1} & -\omega_1 & \cdots & (-1)^{n+1}\omega_1^{2n-3} \\ \frac{1}{\omega_2} & -\omega_2 & \cdots & (-1)^{n+1}\omega_2^{2n-3} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\omega_n} & -\omega_n & \cdots & (-1)^{n+1}\omega_n^{2n-3} \end{bmatrix}^{-1} \begin{bmatrix} D_{11} \\ D_{12} \\ \vdots \\ D_{1n} \end{bmatrix}$$

and

$$\begin{bmatrix} a_2 \\ a_4 \\ \vdots \\ a_{2n} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & \omega_1^2 & \cdots & (-1)^{n+1}\omega_1^{2n-2} \\ -1 & \omega_2^2 & \cdots & (-1)^{n+1}\omega_2^{2n-2} \\ \vdots & \vdots & \ddots & \vdots \\ -1 & \omega_n^2 & \cdots & (-1)^{n+1}\omega_n^{2n-2} \end{bmatrix}^{-1} \begin{bmatrix} D_{21} \\ D_{22} \\ \vdots \\ D_{2n} \end{bmatrix},$$

where D_{ij}, $j = 1, 2$, $i = 1, \cdots, 2n$ are determined by (3.6.1) and (3.6.3).

3.6.2 Constant parametric uncertainty in the presence of periodic external disturbance

In this subsection, it is shown that under appropriate conditions, it is possible to estimate parameter variation from the estimated disturbance even in the presence of periodic disturbances, without modification of the method to estimate the parameter variation introduced in the previous subsection. The analysis is based on the fact that if the disturbances are periodic and the duration of time for integration used for the parameter estimation is long enough, then the effect of the periodic disturbances will almost vanish when integrated.

Assumption 10 *The disturbance/uncertainty in the system consists of some parametric uncertainty and external periodic disturbances.*

In view of Assumption 10, the estimated disturbance $d(t)$ will have the following structure:

$$d(t) = a_1 r_1(t) + \cdots + a_{2n} r_{2n}(t) + \sum_{i=1}^{q} A_i \sin(\bar{\omega}_i t).$$

Suppose $r_1(t)$ tracks the reference signal $R(t) = \sin(\omega t)$ where $\omega \neq \bar{\omega}_i$. Since ω and $\bar{\omega}_i$ are independent with each other, there exists ω such that both

$$\int_0^{\frac{1}{\omega}\pi} A\sin(\bar{\omega}_i(t))\mathrm{d}t \approx 0, \text{ for } i = 1, \cdots, q$$

and

$$\int_{\frac{\pi}{2\omega}}^{\frac{3}{2\omega}\pi} A\sin(\bar{\omega}_i(t))\mathrm{d}t \approx 0, \text{ for } i = 1, \cdots, q$$

hold. If these conditions are satisfied; i.e. $\bar{\omega}_i$ is large enough compared with ω, it is clear that the method of parameter estimation, introduced in Subsection 3.6.1, is not affected by the existence of the external disturbances[2]. Therefore, it is possible to estimate parameter variation from the estimated disturbance in the presence of periodic disturbances.

3.6.3 Constant parametric uncertainty in the presence of constant and periodic disturbance

In this subsection, it is shown that it is always possible to estimate parameter variation caused by parametric uncertainty in the presence of periodic disturbance and constant disturbance under certain assumptions. The method is based on the fact that if a time period of integration, used for estimation of the parameter variation, is appropriate, then it is possible to extract almost exact knowledge of the constant disturbance from the estimated disturbance, which contains the effects of parametric uncertainty, periodic disturbances, and the constant disturbance.

Assumption 11 *The disturbance/uncertainty in the system consists of parametric uncertainty and an external disturbance. The external disturbance is composed of the sum of periodic disturbances and a constant disturbance.*

In view of Assumption 11, the estimated disturbance $d(t)$ has the following structure:

$$d(t) = a_1 r_1(t) + \cdots + a_{2n} r_{2n}(t) + \sum_{i=1}^{q} A_i \sin(\bar{\omega}_i t) + \beta$$

where β is some unknown constant.

As in the previous formulations, consider the situation where $r_1(t)$ tracks the reference signal $R(t) = \sin(\omega t)$. Since $\int_0^{\frac{2\pi}{\omega}} \sin(\omega t)\mathrm{d}t = 0$ and $\int_0^{\frac{2\pi}{\omega}} \cos(\omega t)\mathrm{d}t = 0$, integrating $d(t)$ over the period $0 \leq t \leq \frac{2\pi}{\omega}$ gives

$$\int_0^{\frac{2\pi}{\omega}} d(t)\mathrm{d}t \approx \frac{2\pi}{\omega}\beta. \tag{3.6.5}$$

[2]To be more precise, the effects of periodic disturbances are not exactly zero. However, making ω smaller results their effects almost zero.

Note that it is assumed that ω is small enough with respect to ω_i to eliminate the effect of the periodic disturbances. Using (3.6.5), the constant term of the external disturbance, β, is given by

$$\beta \approx \hat{\beta} := \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} d(t)dt,$$

which can be evaluated numerically.

Next, re-define D_{1i} and D_{2i}, defined at (3.6.1) and (3.6.3) as follows:

$$\begin{aligned} D_{1i} &:= \int_0^{\frac{\pi}{\omega_i}} d(t)dt - \hat{\beta}\frac{\pi}{\omega_i}, \\ D_{2i} &:= \int_{\frac{\pi}{2\omega_i}}^{\frac{3\pi}{2\omega_i}} d(t)dt - \hat{\beta}\frac{\pi}{\omega_i}. \end{aligned}$$

Then, any effects of periodic and constant disturbances are nearly eliminated from D_{ji}. Hence, using the parameter estimation method, introduced in Subsection 3.6.1, it is possible to estimate the parameter variation in the presence of constant and periodic disturbance.

3.6.4 Simulation example

Configuration

The system, to be examined, is a second order single-input linear system expressed as follows:

$$\dot{r}(t) = Ar(t) + B(u(t) + d(t)), \tag{3.6.6}$$

where $d(t)$ is the external disturbance/uncertainty, and the system matrices are given by

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

An initial condition for the system is taken to be $r(t_0) = [0\ 0]^t$. The control input for the tracking is applied to estimate parameter variations. The reference signal for the tracking is chosen as $R(t) = \sin(0.5t)$. The state which tracks the reference signal is $r_1(t)$, where $r_i(\cdot)$ are the components of $r(t) = [r_1(t)\ r_2(t)]^t$. The control input for the tracking, which is represented as $u_{tr}(t)$, is determined as follows. The transfer function of system (3.6.6) between $u_{tr}(t)$ to $r_1(t)$ is given by

$$G(s) = \frac{1}{(s+2)(s+3)}.$$

Hence, the steady state gain with respect to the signal $\sin(0.5t)$ is given by

$$|G(0.5i)| = \frac{1}{\sqrt{39.3125}}.$$

Also, the phase-shift with respect to the signal $\sin(0.5t)$ is given by

$$\begin{aligned} \angle G(0.5i) &= \tan^{-1}(\frac{-2.5}{5.75}) \\ &\approx -0.410127. \end{aligned}$$

Thus, the tracking control input is approximately given by

$$u_{tr}(t) \approx \sqrt{39.3125}\sin(0.5t + 0.410127).$$

For this problem, an accuracy specified by $\epsilon_e^2 = 2.0 \times 10^{-5}$ is required. For the adaptive algorithm, $\delta = -2.0$, $\kappa_2 = 1.5$, and $\omega = 10$ are used and initially, $\dot{\lambda}_1(t_0) = 0$ and $\lambda_1(t_0) = -2.0$ are set. For simulation purposes, the disturbance term is chosen to be $d(t) = a_{v1}r_1(t) + a_{v2}r_2(t) + 0.1\sin t + 0.5\sin(2t) + 0.2\sin(3t) + \beta$ where $a_{v1} = 4.0$, $a_{v2} = 2.0$, and $\beta = 1.0$. The simulation has been performed with the following configuration:

Programing language: C++;

Compiler: g++ version 3.2.2;

Algorithm to obtain numerical solution of ODE: Runge-Kutta (see [31]);

Time step for Runge-Kutta algorithm: 1.0×10^{-4}.

Simulation results

The closed-loop response of the states of the system are shown in Figures 3.6.1 and 3.6.2. The state to be controlled and the reference signal are shown in Figure 3.6.1. The solid line and dashed line represent the state to be controlled and the reference signal, respectively. In this figure, it is observed that the state converges to the reference signal very rapidly. The actual and estimated disturbances are shown in Figure 3.6.3. The solid line and the dashed line represent the actual and estimated disturbances, respectively. In this figure, it is observed that the estimated disturbance converges to the actual one very rapidly.

Estimated parameter variations and constant disturbance are shown at Figure 3.6.4. The solid line segment represents the estimate of parameter variation a_{v1}, the dashed line segment represents the estimate of parameter variation a_{v2}, and the dotted line segment represents the estimate of constant disturbance β. The actual values are $a_{v1} = 4.0$, $a_{v2} = 2.0$, and $\beta = 1.0$ respectively. From Figure 3.6.4, it is seen that the estimation errors are very small. Therefore, the simulation confirms that the performance of the method for estimation of parameter variation is good.

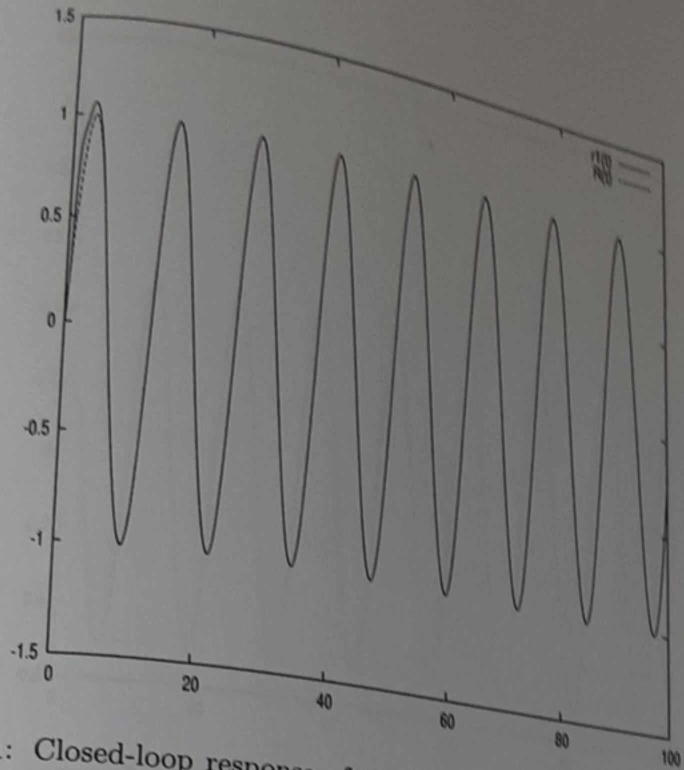

Figure 3.6.1: Closed-loop response of the state $r_1(t)$ and the reference signal $R(t)$.

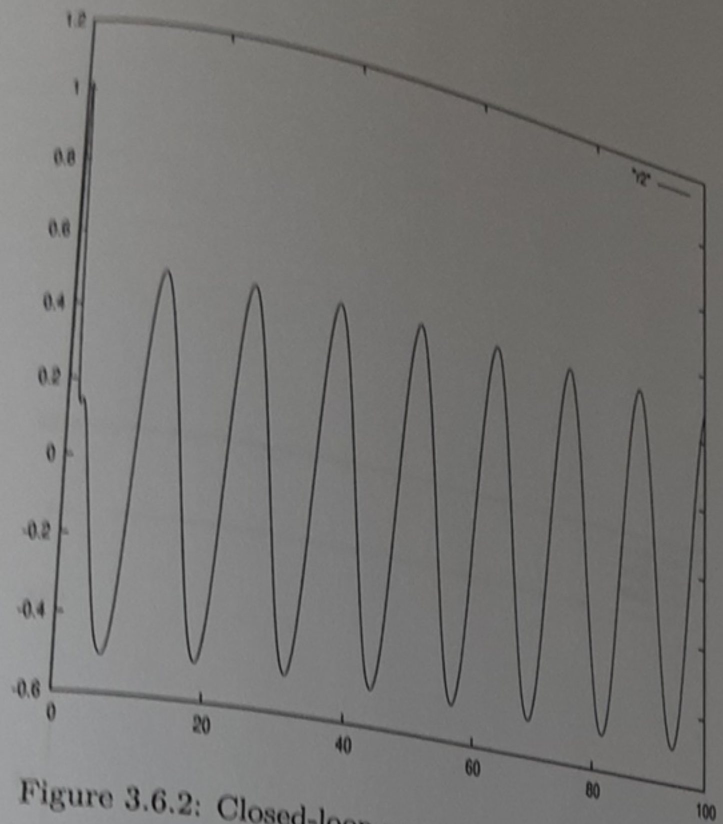

Figure 3.6.2: Closed-loop response of the state $r_2(t)$.

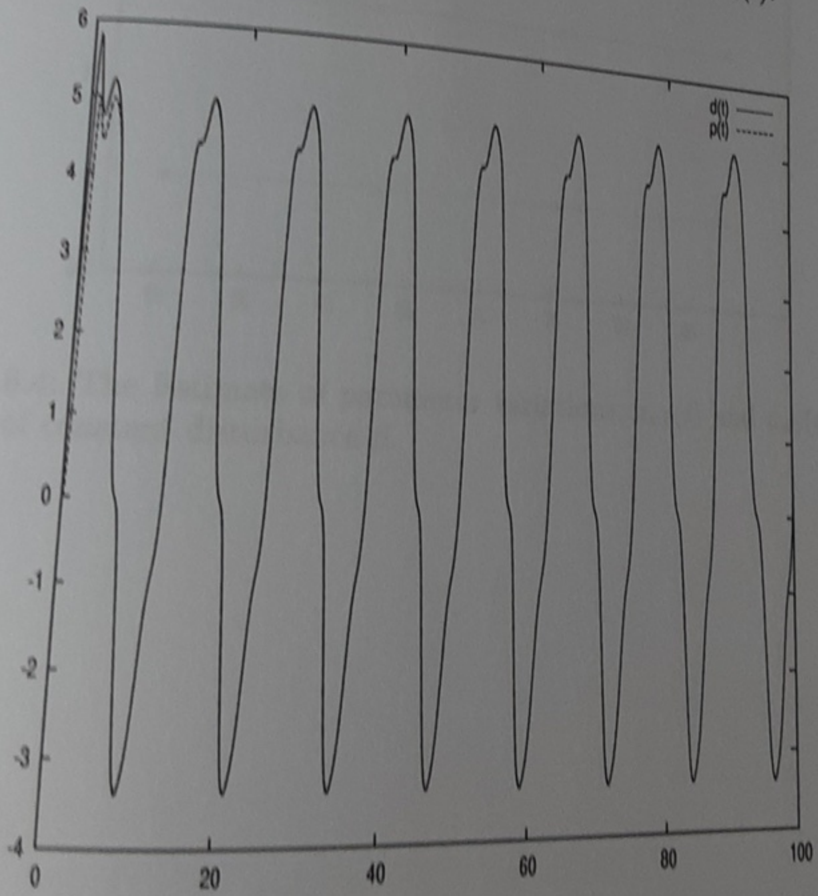

Figure 3.6.3: The actual and estimated disturbances: $d(t)$ and $p(t)$, respectively.

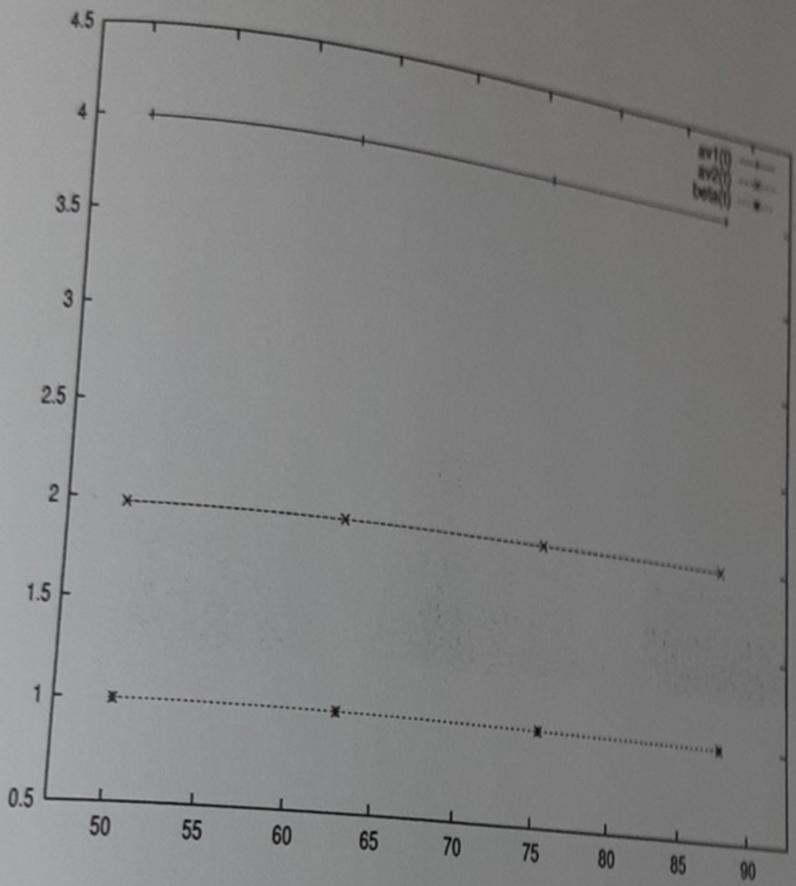

Figure 3.6.4: The Estimate of parameter variations, $a_{v1}(t)$ and $a_{v2}(t)$ and the estimate of constant disturbance β.

3.7 Conclusions

In this chapter, applications of the method of disturbance estimation and cancellation are investigated. Investigated topics include robust properties of the system for both stabilisation and tracking problems, input uncertainty and unmodelled dynamics, estimation and cancellation of residual uncertainty/disturbance, estimation and cancellation by output measurement, and extraction of parameter variation from estimated disturbance. As a consequence of these investigations, it is expected that, in practical situations, the method is rigorous, is easy to implement, and results beneficial information.

Chapter 4

Conclusions and further research

4.1 Concluding remarks

As it is shown in previous chapters, in this study, the following topics are investigated.

1. For both single-input and multi-input systems, estimation and cancellation of bounded disturbance/uncertainty can be performed without *a priori* knowledge of bounded disturbance/uncertainty (see Theorem 3, 4, 5, and related remarks).

2. Estimation and cancellation can be achieved, even in the presence of residual uncertainty/disturbance under appropriate conditions (see Section 3.4).

3. Using the disturbance estimation method, a tracking controller can be designed with respect to the nominal model (see Theorem 7).

4. Estimation and cancellation of unknown bounded disturbance/uncertainty can, also, be achieved by using only output measurement, even in the presence of sensor noise (see Section 3.5).

5. For both the stabilization and tracking problems, the system to be controlled is robust against parametric uncertainty, input uncertainty, unmodelled dynamics, external disturbance and/or sensor noise using the disturbance estimation/cancellation method (see Theorem 7, 8, and Section 3.5.3).

6. The parameter variations for the nominal model can be estimated from the estimated disturbance under certain assumptions (see Section 3.6).

In addition, numerical simulations are presented to demonstrate the methods developed.

4.2 Recommendations for further work

Topics which are not studied, but are suggested for further work, are listed as follows.

1. Implementation of the methods in some practical application.
2. Treatment of the case when there are input constraints.

The author believes that theory is enhanced as a result of interaction with various applications. Although the primary motivation of this study comes from one specific area of application and the assumptions developed are constructed so that these assumptions are realistic in the specific practical situation, it is uncertain what kind of problems exist when the theory is implemented for applications. The author believes that almost exact estimation of such uncertainty can be achieved but perfect estimation of such uncertainty is not possible. Thus, to make clear and resolve such problems, it is recommended that the methods be implemented in a number of applications.

In relation to *Treatment of input constraint*, improvements of the adaptive algorithms and a better understanding of the nominal model and modelling is encouraged. The method proposed gives an engineering solution for the worst case situation regarding uncertainty and disturbance, as well as control of a nonlinear and time-varying system, by cancelling out their effect. However, due to input constraints, there is a limit on the amount of estimation error of uncertainty and uncertainty that can be tolerated as modelling errors. Thus, improvements of the adaptive algorithms and a better understanding of modelling and the characteristics of the nominal model, which are fundamental characteristics of nonlinear system and time-varying system, are required.

Since the methods developed do not require any *a priori* knowledge of disturbance and the methods can be applied using only output measurement, the methods will be relatively easy to implement in a practical situation. The author believes that the method proposed will contribute to the development of high performance/reliable systems.

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